Complex Numbers and Colors

This is the tenth anniversary edition of the calendar *Complex Beauties*. To celebrate, we took the liberty of altering the phase portrait of the title picture to enhance special properties of the function and to improve the appearance of the image. Apart from that, we follow our established procedure: Each month we present a special function. On the front page, we show the phase portrait and on the back page we provide an exposition of the mathematical background. We try to make these explanations accessible to the general public in order to provide a glimpse of some topics from classical and modern mathematics. In some cases this is quite challenging. Even without this background, it is still possible to admire the images to get an impression of the magic of the world of mathematical structures. We also include biographical sketches of the mathematicians whose work has contributed to the understanding of the functions presented.

The construction of phase portraits is based on the interpretation of complex numbers \( z \) as points in the Gaussian plane. The horizontal coordinate \( x \) of the point representing \( z \) is called the real part of \( z \) (\( \text{Re} z \)) and the vertical coordinate \( y \) of the point representing \( z \) is called the imaginary part of \( z \) (\( \text{Im} z \)); we write \( z = x + iy \). Alternatively, the point representing \( z \) can also be given by its distance from the origin (\( |z| \), the modulus of \( z \)) and an angle (\( \arg z \), the argument of \( z \)).

The phase portrait of a complex function \( f \) (appearing in the picture on the left) arises when all points \( z \) of the domain of \( f \) are colored according to the argument (or “phase”) of the value \( w = f(z) \). More precisely, in the first step the color wheel is used to assign colors to the complex \( w \)-plane: points on rays emanating from the origin are assigned the same color (as in the picture on the right). Thus, points with the same argument are assigned the same color. In the second step, every point \( z \) in the domain of \( f \) is assigned the same color as the value \( f(z) \) in the \( w \)-plane.

The phase portrait can be thought of as the fingerprint of the function. Although only one part of the data is encoded (the argument) and another part is suppressed (the modulus), an important class of functions (“analytic” or, more generally, “meromorphic” functions) can be reconstructed uniquely up to normalization. Certain modifications of the color coding allow us to see properties of the function more easily. In this calendar we mainly use three different coloring schemes: the phase portrait as described above and two variations shown in the second row of pictures. The variation on the left adds the modulus of the function to the representation; the version on the right highlights, in addition, preservation of angles under the mapping.

An introduction to function theory illustrated with phase portraits can be found in E. Wegert, *Visual Complex Functions – An Introduction with Phase Portraits*, Springer Basel 2012. Further information about the calendar (including previous years) and the book is available at

www.mathcalendar.net, \hspace{1em} www.visual.wegert.com.

We thank all our faithful readers and the Verein der Freunde und Förderer der TU Bergakademie Freiberg e. V. for their valuable support of this project.
Hermite Polynomials

Hermite polynomials were defined in 1810 by Pierre-Simon Laplace and studied by Pafnuty Chebyshev in 1859. In 1864, Hermite wrote about them in his paper *Sur un nouveau développement en série de fonctions*. Though these polynomials were not new, Hermite was the first to consider multidimensional polynomials.

The $n$th-degree Hermite polynomial is defined by

$$ He_n(x) = (-1)^n e^{x^2/2} d^n e^{-x^2/2} dx^n. $$

These are important in both probability and physics, though the definition is modified in the latter case to be: $H_n(x) = (-1)^n e^{x^2} d^n e^{-x^2} dx^n$. These two definitions are related via scaling: $H_n(x) = 2^{n/2} He_n(\sqrt{2}x)$. In the “probabilist’s version” we see the appearance of the probability density function for the standard normal distribution, $e^{-x^2/2}/\sqrt{2\pi}$. The picture on the right shows the graph of the functions $He_n$ for $n = 0, 1, 2, 3, 4$.

The Hermite polynomials are orthogonal with respect to an inner product with weight function given by the standard probability density function; i.e.,

$$ \langle H_m(x), H_n(x) \rangle = \int_{-\infty}^{\infty} He_m(x) He_n(x) e^{-x^2/2} dx = 0 $$

for $m \neq n$ and when $m = n$, this value is $\sqrt{2\pi n!}$. In fact, the Hermite polynomials form an orthogonal basis for the Hilbert space consisting of functions $f$ for which $\int_{-\infty}^{\infty} |f(x)|^2 e^{-x^2/2} dx < \infty$. These polynomials are important in the solution of Hermite’s differential equation:

$$ u'' - xu' + \lambda u = 0, $$

where $\lambda$ is a constant; this has a non-zero solution (which does not grow faster than a polynomial at infinity) if and only if $\lambda$ is a nonnegative integer. The solution is then essentially unique, namely, every solution is a constant multiple of $He_\lambda(x)$. In this month’s picture we see the function $w(z) = e^{-z^2/2} He_{10}(z)$.

Charles Hermite (1822 – 1901)

was one of seven children. He was born with a deformed right foot, which would play an important role in his life. He attended the Collège de Nancy, and then the Collège Henri and Collège Louis-le-Grand in Paris. He entered École Polytechnique, but remained only a year; he was not permitted to continue his studies, due to his deformed foot. Though this decision was eventually reversed, conditions that Hermite was unable to accept were imposed. In 1842, Hermite’s proof of Abel’s theorem concerning the impossibility of solving quintic equations was published. Hermite began a correspondence with Jacobi and he befriended Joseph Bertrand, whose sister Hermite married. He was appointed to the École Polytechnique in 1848 as admissions examiner and répétiteur and as maître de conférence in 1862. Hermite was elected to the Académie des Sciences in 1856, the same year that he contracted smallpox.

Hermite is known for his contributions to many fields, including number theory, algebra, elliptic functions, and orthogonal polynomials. Hermite polynomials, Hermite interpolation, Hermitian matrices, Rue Charles Hermite, Square Charles Hermite in the 18th Arrondissement in Paris, and the lunar impact crater, Hermite, are all named in his honor.

Brower’s Fixed Point Theorem for Blaschke Products

Suppose a function \( f : X \to X \) maps a set to itself. If for some \( x \in X \) we have \( f(x) = x \) then we say that \( x \) is a fixed point of \( f \). Fixed points and theorems about them are very useful in pure and applied mathematics. Perhaps the most basic such theorem is Brouwer’s fixed point theorem. A celebrated consequence of this theorem is the existence of a Nash equilibrium in game theory, which means that multi-player games have solutions that are optimal for all players.

With this month’s title picture we will illustrate Brouwer’s theorem applied to a Blaschke product (see March 2012 for Blaschke products). In two dimensions, Brouwer’s fixed point theorem simply says that a continuous function from a closed, bounded, and convex set to itself has a fixed point. The theorem has many extensions to higher dimensions and to more general functions.

We consider the following Blaschke product \( B \) of degree 6:

\[
B(z) = \frac{(z - 0.9i)(z + 0.9i)(z - 0.9)(z - 0.9 \exp(2\pi i/3))(z - 0.9 \exp(4\pi i/3))}{(1 + 0.9iz)(1 - 0.9iz)(1 - 0.9z)^2(1 - 0.9 \exp(2\pi i/3)z)(1 - 0.9 \exp(4\pi i/3)z)}.
\]

This function maps the closed unit disk \( \overline{D} \) onto itself, and it is continuous on this set since the denominator has all of its zeros outside the closed disk. Thus, Brouwer’s fixed point theorem applies. In fact, this particular function has five fixed points on the circle, one inside the disk and one outside the disk. The fixed point outside the disk is the reciprocal of the complex conjugate of the fixed point inside the disk because for every Blaschke product \( b \) we have \( b(1/z) = 1/b(z) \). The figure on the left shows the phase portrait of \( B \). Note the zero of order 2 at 0.9. The figure on the right shows \( B(z) - z \) on a region large enough to show the fixed point outside the disk. The existence of the fixed point outside the closed disk is a consequence of our Blaschke product having a fixed point in the open disk rather than of Brouwer’s theorem.

Brouwer’s fixed point theorem does not guarantee a fixed point inside the disk for every Blaschke product. It is possible that all fixed points lie on the circle. An example of such a Blaschke product is presented in May of this calendar when we discuss the Denjoy-Wolff point.

This month’s title page shows the phase portrait of \( B_3(z) - z \), where \( B_3 = B \circ B \circ B \) denotes the third iteration of the Blaschke product defined above. Taking the third iteration keeps all the fixed points of \( B \), but the degree 216 Blaschke product provides a more interesting picture.

**L. E. J. Brouwer (1881 – 1966)**

who was known as Bertus by his family and friends, was born in Overschie, the Netherlands. He studied mathematics at the University of Amsterdam but was also very interested in philosophy. Even before finishing his Ph.D. thesis, he published the treatise *Life, Art, and Mysticism*.

Brouwer did groundbreaking work in point set topology, but he was perhaps even more interested in the philosophical foundations of mathematics. In 1909, he obtained his first appointment at the University of Amsterdam. When his thesis advisor, Korteweg, retired, Brouwer was appointed his successor, highly recommended by Hilbert. Despite offers from Berlin and Moscow, Brouwer stayed in Amsterdam until his retirement in 1951. He married in 1905, but the couple did not have children.

After 1913, Brouwer worked and lectured mostly on the foundations of mathematics representing mathematical intuitionism. In this philosophy, proofs can rely on self-evident logical statements only. In particular, the principle of excluded middle is not accepted, which has the consequence that existence proofs cannot be done by proving the impossibility of non-existence. Thus, intuitionism relies on constructive proofs only. His theories were not widely accepted among mathematicians and he had to endure many controversies. Despite that, he was elected to several academies and received many honors. There has been a recent resurgence of his theory in computer science.

After giving lectures abroad at old age, he died in Blaricum near Amsterdam, in a traffic accident in front of his house.

J J O’Connor and E F Robertson, L.E.J. Brouwer, MacTutor History of Mathematics, (University of St Andrews, Scotland, October 2003)
The Riesz Function

For complex numbers $s$ with $\Re s > 1$, the series

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \sum_{n=1}^{\infty} e^{-s \log n}$$

converges and it can be extended analytically to the complex plane except for $s = 1$. This function is called the Riemann zeta function. The Riemann hypothesis (see November, 2011) conjectures that the Riemann zeta function has its zeros only at complex numbers with real part $1/2$ and at the negative even integers. The inset picture shows this function in the window $-40 \leq \Re s \leq 10$ and $-2 \leq \Im s \leq 48$. We see the pole at $z = 1$, the zeros on the negative real axis, and other zeros on the critical line $\Re z = 1/2$. The Riemann hypothesis is important, in part, for what it tells us about the distribution of prime numbers.

The Riesz function, denoted Riesz, is defined by

$$\text{Riesz}(x) = \sum_{k=1}^{\infty} \frac{(-1)^{k+1} x^k}{(k-1)! \zeta(2k)}.$$ 

In 1916, Marcel Riesz showed that the statement that for all $\delta > 1/4$ there is a real number $M$ such that $|\text{Riesz}(x)| \leq Mx^\delta$ for all sufficiently large $x$ is equivalent to the statement that the Riemann hypothesis holds. It is known that $\zeta(2n) \to 1$ as $n \to \infty$ so the radius of convergence of the Riesz function is infinite and, extended to the complex plane, this defines an entire function. There are other expressions for the Riesz function; in this month’s picture to represent the Riesz function, we used the window $-120 \leq \Re z \leq 280, -200 \leq \Im z \leq 200$ and the form given below involving the Möbius function $\mu(n)$, a function presented in 1832 by August Ferdinand Möbius. For a positive integer $n$, this function is defined to be $0$ if $n$ has one or more repeated prime factors, $1$ if $n = 1$, and $(-1)^k$ if $n$ is a product of $k$ distinct primes. With this notation we have

$$\text{Riesz}(z) = z \left(\frac{6}{\pi^2} + \sum_{n=1}^{\infty} \frac{\mu(n)}{n^2} \left(e^{-z/n^2} - 1\right)\right).$$

Marcel Riesz (1886 – 1969)

was born in Hungary. He studied in Budapest, Göttingen and Paris. He was invited to Sweden in 1908 and remained there until his retirement from Lund University. His students in Stockholm included Olof Thorin, Harald Cramér and Einar Hille; his students in Lund included Otto Frostman and Lars Hörmander. His work first focused on summability of power series, trigonometric series, and Dirichlet series. After moving to Lund, Riesz began working in potential theory and wave propagation, and he became interested in number theory. He wrote one very influential paper with his brother Frigyes (Frederick). In Marcel Riesz in Memoriam, David Rowe writes, “Riesz loved to talk about mathematics and he appreciated having listeners. He could go on for hours and when he was in good form, his grip on the listener never slackened.” After retiring in 1952, Riesz spent many years in the United States visiting several universities (Chicago and Maryland, in particular). In 1962, he had a breakdown and returned to Lund. He was elected to the Swedish Academy of Sciences, the Royal Physiographic Society in Lund, and the Royal Danish Academy of Sciences and Letters (Videnska Selskab).

The Gibbs Phenomenon

To a $2\pi$-periodic function $f$ on $\mathbb{R}$, we may associate a Fourier series of the form

$$a_0 + \sum_{n=1}^{\infty} a_n \cos(nx) + b_n \sin(nx), \quad a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \cos(nt) \, dt, \quad b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \sin(nt) \, dt.$$ 

The coefficients $a_n, b_n$ exist even under very weak conditions, like the integrability of $f$ over $[-\pi, \pi]$. However, the question of whether the series converges to $f$ is difficult to answer and was a major driver of advances in analysis in the 19th century. As an illustration, consider the function $f(x) = x/2$ for $-\pi < x < \pi$ extended to $\mathbb{R}$ by $f(x + 2\pi) = f(x)$ and $f((2n + 1)\pi) = 0$ for all integers $n$. The $n$-th partial sum of the associated Fourier series is then

$$s_n(x) = \sum_{k=1}^{n} (-1)^{k+1} \frac{\sin(kx)}{k}.$$ 

Near jump discontinuities of $f$ we observe an overshoot of $s_n$ that does not die out as $n$ increases. The largest value below $\pi$ at which we have a maximum of $s_n$ is $x_n = n\pi/(n+1)$ and we may calculate

$$\lim_{n \to \infty} s_n(x_n) = \int_{0}^{\pi} \frac{\sin t}{t} \, dt = \frac{\pi}{2} \cdot 1.17898\ldots,$$

which is about 18% larger than the expected value of $\pi/2$.

In 1898/99, in a discussion in the journal *Nature* about Albert Michelson’s harmonic analyzer, Josiah Willard Gibbs pointed out the difference between the graph of the limiting function, $f$, and the set $L$, that is the limit of the graphs of the partial sums $s_n$. In our example, the sums $s_n$ approach the function $f$ pointwise (the blue picture). In contrast, the graphs of the $s_n$ approach the red set, $L$. That is, each point of $L$ is the limit point of a sequence with its $n$-th point in the graph of $s_n$. In 1906, Maxime Bôcher did the first mathematical analysis of this effect and coined the term *Gibbs phenomenon*. – He did this without knowing that this phenomenon had already been described in a paper by the English mathematician Henry Wilbraham in 1848.

The picture of the month is inspired by a complex version of the Gibbs phenomenon that is based on the convergence property of the Taylor series of $f(z) = \log((1 + z)/(1 - z))$ at $z = 1$.

Josiah Willard Gibbs (1839 – 1903)

was born in New Haven, Connecticut, into a family of academics and clergy. In his hometown, he studied mathematics and natural sciences earning the first doctorate in engineering that was awarded in the United States. From 1866 to 1869, together with his sister, he traveled through Europe. He attended lectures by Liouville (see December), among others, in Paris, by Weierstrass and Kronecker in Berlin, and Kirchhoff, Helmholtz and Bunsen in Heidelberg. Returning to Yale University in New Haven, he was named the first American professor of mathematical physics in 1871. It was an unsalaried position until 1880, but due to his inheritance he was financially independent.

His first publication did not appear until 1873, but it was enthusiastically received by Maxwell. After that, Gibbs published a sequence of articles in thermodynamics that made him one of the founders of physical chemistry. Among these papers was a description of Gibbs phase rule. He is also one of the founders of statistical mechanics, together with Maxwell and Boltzmann. In addition, he contributed to the electromagnetic theory of light. He is one of the people who introduced modern vector analysis (scalar and cross product) that simplified calculations previously done using the prevalent quaternion calculus.

May

Su  Mo  Tu  We  Th  Fr  Sa  Su  Mo  Tu  We  Th  Fr  Sa
1   2   3   4   5   6   7   8   9
10  11  12  13  14  15  16  17  18  19  20  21  22  23
24  25  26  27  28  29  30  31
The Denjoy-Wolff Point of a Blaschke Product

In 1922 Arnaud Denjoy, a French mathematician, and Julius Wolff from the Netherlands collaborated while both of them were at the University of Utrecht. Only four years later, in 1926, each of them independently proved a limit theorem about iterations of certain holomorphic functions of the unit disk. More precisely, they showed that if \( \varphi \) is a holomorphic map of the unit disk \( D \) into itself other than an automorphism of \( D \) with an interior fixed point, then there is a point \( \omega \) in \( D \) so that the iterates \( \varphi_n \) converge to \( \omega \) uniformly on compact subsets of \( D \). This theorem is now referred to as the Denjoy-Wolff theorem and the point \( \omega \) is the Denjoy-Wolff point of the map \( \varphi \).

In this month’s title picture, this theorem is illustrated with a Blaschke product \( B \) of degree 2 (see March 2012 for Blaschke products). The chosen function has zeros at \( e^{5\pi i/6}/\sqrt{3} \) and \( e^{-\pi i/6}/\sqrt{3} \). Its phase portrait is shown in the first of the two inset pictures. The second picture shows the phase portrait of \( B(z) - z \) and from it we can see that our Blaschke product has no fixed point in \( D \). In fact, the only zero of \( B(z) - z \) is of order 3 at \( e^{\pi i/3} \). By the Denjoy-Wolff theorem we can conclude that the iterates of \( B \) will converge (uniformly on compact subsets of \( D \)) to its Denjoy-Wolff point on \( T \). The phase portrait on the front page is the 8th iteration of \( B \), where the convergence is already quite good.

Any iterate of a Blaschke product is again a Blaschke product. Also, for \( |z| > 1 \) we find that \( C(z) = 1/C(1/z) \), where \( C \) is any finite Blaschke product. Thus, outside the unit circle the iterates converge to a point as well and, in our case, it is the same point as the one approached in \( D \) and it is in fact the fixed point \( e^{\pi i/3} \) of \( B \). However, on \( T \) there is, in general, no convergence: If \( n \) is the number of iterations of the degree 2 Blaschke product \( B \), then \( B^n \) winds the unit circle \( 2^n \) times around itself, taking on each value of \( T \) exactly \( 2^n \) times. This can also be seen on the portrait of the month.

In February, we considered a Blaschke product with an interior fixed point in \( D \). It is straightforward to show that in such a case the Denjoy-Wolff point has to be the fixed point. The picture on the right shows the eighth iteration of this function. Indeed, the phase portrait is almost completely red, which is the color of the positive real axis. This agrees with the observation that the fixed point in February’s picture is positive real.

Arnaud Denjoy (1884 – 1974)

was born in Auch, France, the son of a wine merchant and a Catalanian mother. He started secondary school in Auch and finished it in Montpellier before going to Paris to study at the École Normal Supérieur. Among his professors were Borel, Painlevé and Picard. In 1909 he finished his dissertation under the guidance of René-Louis Baire. He held positions at the Université de Montpellier and Universiteit Utrecht before taking on a professorship at the Université de Paris in 1922, where he stayed until his retirement in 1955. He had several students, among them Jacqueline Ferrand (see October), Gustave Choquet and Charles Pisot.

Arnaud Denjoy married Thérèse-Marie Chevresson and the couple raised three sons. Arnaud had many interests outside mathematics. He was politically active as a member of the Radical Socialist Party, serving as town councillor of Montpellier, he was interested in philosophy and social studies, and he enjoyed walking and cycling in the back country.

He did very influential mathematical work in real and harmonic analysis, integration theory, and differential equations on the torus that led to dynamical systems. Among many other honors he was elected to the Académie des sciences in 1942 and was its president in 1962.

The Poisson Distribution

In observing the decay of small amounts of a radioactive substance in a time interval that is much smaller than the half-life, one may assume that (i) the probability of one atom decaying is proportional to the length of the time interval (constant rate of decay) and (ii) that two different decays do not affect each other (events are independent). Random variables with the properties (i) and (ii) are referred to as Poisson processes.

If the time interval is fixed and \( \lambda \) denotes the expected number of events (decays) during this time interval, then the probability that exactly \( k \) events will occur is

\[
P_\lambda(k) = \frac{\lambda^k}{k!} e^{-\lambda}.
\]

The first inset picture shows this discrete probability density function for the parameters \( \lambda = 5 \) (red) and \( \lambda = 10 \) (blue).

Summing the probabilities \( P_\lambda(k) \) for \( k = 0, \ldots, n \) one obtains the cumulative density function (cdf)

\[
F_\lambda(n) = e^{-\lambda} \sum_{k=0}^{n} \frac{\lambda^k}{k!}.
\]

This is the cdf of the Poisson distribution and denotes the probability that at most \( n \) events will occur in the fixed time interval. One may consider this function for fixed \( n \) as a function of the variable \( \lambda \); that is, as the probability of at most \( n \) events occurring dependent on the rate \( \lambda \). The second inset picture shows the graph of the function \( \lambda \to F_\lambda(10) \).

The picture of the month shows the phase portrait of this function with complex values: \( z \to F_z(10) \). It is the quotient of the 10-th partial sum of the Taylor series of \( e^z \) and \( e^z \) itself. The derivative of this function has a zero of order 10 at \( z = 0 \). This is reflected in the very flat behavior of the function \( \lambda \to F_\lambda(10) \) for small real \( \lambda \).

Siméon Denis Poisson (1781 – 1840)

was a frail child and several of his siblings had died. His father, determined for his son to have a good future, wanted Poisson to become a physician. Because Poisson had no interest in this profession, he stopped his medical education and, in 1796, started to study at the École centrale in Fontainebleau. Two years later he changed to the École Polytechnique in Paris, where he got to know Laplace and Lagrange. Two years after he completed his studies, in 1802, he was appointed professor. In 1806, he was the successor to the chair held by Joseph Fourier.

Poisson published approximately 350 mathematical papers. He studied the foundations of the physics of waves, he worked in acoustics, elasticity and heat as well as in electrical properties of solids. He first introduced and published work about the Poisson distribution in 1837 in his paper “Research on the probability of judgments in criminal and civil matters” (in French).

Poisson’s work was largely ignored by French mathematicians, but he received great recognition from abroad. In 1812, Poisson became a corresponding member of the Prussian Academy of Sciences and, from 1830 on, a foreign member. In 1818, he became a fellow of the Royal Society and, in 1822, he was elected to the American Academy of Arts and Sciences. In December 1826, he became an honorary member of the Russian Academy of Sciences in Saint Petersburg.

Poisson’s name is immortalized in the Eiffel tower and it is connected to a great number of concepts and ideas: the Poisson integral, the Poisson equation in potential theory, the Poisson brackets in differential equations, the Poisson transformation, the Poisson quotient in elasticity, and the Poisson constant in electricity. A lunar crater and the asteroid 12874 are named after him as well.
Fabry’s Gap Theorem

Every power series has a radius and disk of convergence (though we note that the radius of convergence might be 0 or infinite). Some functions can be continued analytically beyond this region. If we consider, for example, the function $g_1(z) = \sum_{k=0}^{\infty} (-1)^k(z - 1)^k$, the disk of convergence can be shown to be $S_1 := \{z : |z - 1| < 1\}$. This series diverges at $z = 0$, which lies on the boundary of the circle. If we consider

$$g_2(z) = \frac{2}{3} \sum_{k=0}^{\infty} \left(1 - \frac{2z}{3}\right)^k,$$

this has disk of convergence $S_2 := \{|z - 3/2| < 3/2\}$. Using what we know about geometric series, we see that these both define the function $F(z) = 1/z$, but the set $S_2$ is larger than the set $S_1$ – though neither set contains the point 0. Below we see the functions $g_1, g_2$ (approximated by the 100th partial sums), and $F$ in the region $-1 \leq \text{Re}z \leq 3$ and $-2 \leq \text{Im}z \leq 2$.

In this month’s picture, we focus on functions with disk of convergence the unit disk that illustrate a theorem Fabry proved in 1896. Fabry’s gap theorem states that if $f(z) = \sum a_kz^{n_k}$ is a power series with the unit disk as the disk of convergence and $\lim_{k \to \infty} n_k/k = \infty$, then the unit circle is the natural boundary of $f$; that is, unlike the previous example, the power series cannot be continued analytically past the circle. In fact, every point of the circle is a singularity of $f$. We see this in the inset picture below, which is a close-up of this month’s picture. The “gap” in the title refers to the gap between the exponents. The function featured here has power series $\sum_k k^2 z^k$, which satisfies the conditions of Fabry’s theorem. Looking at the picture in the inset, we see that (as we noted before) each point of the unit circle is a singularity.

In 1929, Pólya proved a converse of this result, namely if $(n_k)$ is a sequence of integers such that the $\lim \inf n_k/k$ is finite, then there is a power series $\sum a_kz^{n_k}$ with the unit disk as disk of convergence for which the unit circle is not the natural boundary.

Eugène Fabry (1856 – 1944)

was born in Marseille, the second child in a family of five children. The children each grew up to be well known in their professions: The oldest, Auguste, was a lawyer; he was followed by Eugène (a mathematician), then Louis (an astronomer), Charles (a physicist, specializing in optics), and Pierre (an engineer). Eugène Fabry studied in Marseille, and he received an engineering degree in 1876. In 1878, he received his licentiate in physics. He first worked as an engineer and, in 1885, he received his doctorate in mathematics. He was a maître de conférence in the faculties of sciences at Rennes and Nancy, professor in the faculty of sciences of Montpellier and later Marseille, examiner at the École Polytechnique, member of the Académie des Sciences et Lettres de Montpellier, and contributor to the French edition of the Encyclopedia of Mathematical Sciences.

Green’s Function

In many areas of physics harmonic functions, which are solutions of the Laplace equation \( \Delta U = 0 \), are of great importance. Here the Laplace operator \( \Delta \) in \( n \)-space with coordinates \( x = (x_1, \ldots, x_n) \) is defined by

\[
\Delta U = \frac{\partial^2 U}{\partial x_1^2} + \frac{\partial^2 U}{\partial x_2^2} + \cdots + \frac{\partial^2 U}{\partial x_n^2}.
\]

The investigation of harmonic functions is the subject of potential theory.

In two dimensions there is a close relation between harmonic and analytic functions: The real part \( u \) and the imaginary part \( v \) of an analytic function \( w = u + iv \) in the variable \( z = x + iy \) are mutually conjugate harmonic functions in the variables \( x \) and \( y \). As an example, consider a two-dimensional electric field. In empty space this may be described by a potential \( U \), which is a harmonic function. The level curves, \( U = \text{const} \), are called equipotential lines. The curves perpendicular to them are the field lines. The inset picture on the left is an illustration of a field generated by a point charge in a circular disk that is grounded at the edge.

George Green developed a method to solve the Poisson equation \( \Delta U = f \) with boundary values:

\[
\Delta U(x) = f(x), \quad x \in D, \quad U(x) = 0, \quad x \in \partial D.
\]

His method allowed for a continuous superposition of potentials generated by point charges. To this end, he introduced a function \( G = G(x, y) \) that describes the potential at a point \( x \) generated by a point charge at \( y \) and that also satisfies the boundary conditions. The solution of the boundary value problem is then given by

\[
u(x) = \int_D G(x, y) f(y) \, dy.
\]

This month’s picture is a visualization of Green’s function as a phase portrait in a region that is complementary to four grounded circular disks. The following technique may be used: If the potential \( U \) is known, then we can construct the conjugate harmonic function \( V \) and \( W = U + iV \) is analytic. The function \( w = e^{iW} = e^{-V} e^{iU} \) has argument \( U \); hence, the isochromatic lines of \( w \) are the equipotential lines of \( U \). However, the function \( V \) that is conjugate harmonic to \( U \) exists on multiply connected regions \( D \) of a Riemann surface. Thus, the function presented here has branch cuts across which the modulus is discontinuous. Since the argument is continuous, these discontinuities are not visible in the chosen representation. The inset picture on the right shows the function in a coloring scheme that includes the modulus.

George Green (1793 – 1841)

was the son of a baker in Sneinton, near Nottingham. His elementary education lasted only one year; from an early age he had to work in the bakery and in the mill that his father built in 1807. Despite these bothersome jobs, he found the time to educate himself in mathematics, reaching the forefront of his time. In 1828, he self-published An Essay on the Application of Mathematical Analysis to the Theories of Electricity and Magnetism, in which we find Green’s theorem and Green’s function. At first, this work was hardly noticed – only Edward Bromhead started to support him. As a consequence, Green was able to start studying in Cambridge at the age of 40, a study he very successfully completed in 1837. During his short academic career he published in acoustics, optics, and hydrodynamics. In 1840, he returned sick to Sneinton and died the following year. The mathematical significance of his work was appreciated only posthumously, mainly through the promotion by William Thomson. According to the curator of an exhibition about Green at the University of Nottingham, “there are no known portraits or photos” of Green. We therefore show his mill that is now a museum.

The function on this month’s front page was computed using Matlab code from L.N. Trefethen, Series Solution of Laplace Problems, Anziam J. 60 (2018) 1–26.
September

<table>
<thead>
<tr>
<th>Su</th>
<th>Mo</th>
<th>Tu</th>
<th>We</th>
<th>Th</th>
<th>Fr</th>
<th>Sa</th>
<th>Su</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>6</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>7</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>8</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>9</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>10</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>11</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>12</td>
</tr>
<tr>
<td>13</td>
<td>14</td>
<td>15</td>
<td>16</td>
<td>17</td>
<td>18</td>
<td>19</td>
<td>20</td>
</tr>
<tr>
<td>21</td>
<td>22</td>
<td>23</td>
<td>24</td>
<td>25</td>
<td>26</td>
<td></td>
<td></td>
</tr>
<tr>
<td>27</td>
<td>28</td>
<td>29</td>
<td>30</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
The Kolmogorov-Smirnov Test

The Kolmogorov-Smirnov test is a nonparametric statistical test to determine whether a random variable has a conjectured probability distribution.

To do so, we start with a random sample of \( n \) values for the random variable to be tested and calculate the empirical cumulative distribution function (cdf), \( F_n \), for these data. The random variable \( K_n \) denotes the distance of this function to the (continuous) cdf, \( F_0 \), of the assumed distribution (using the supremum norm) multiplied by \( \sqrt{n} \). Surprisingly, the distribution of \( K_n \) is independent of \( F_0 \) and as \( n \to \infty \), the random variable \( K_n \) tends towards a random variable \( K \) with what is called the Kolmogorov distribution.

Its cdf has the series expansion
\[
P(K \leq x) = 1 + 2 \sum_{k=1}^{\infty} (-1)^k e^{-2k^2 x^2}, \quad x > 0.
\]

The null hypothesis of the Kolmogorov-Smirnov test is the assumption that the random variable in question has the distribution described by \( F_0 \). This hypothesis will be rejected if the probability of the empirical distance function \( K_n \) surpasses a threshold. This threshold can be determined with the help of the Kolmogorov distribution.

The series (1) has an interesting convergence property: Even though we have pointwise convergence for all \( x > 0 \), the partial sums \( s_n \) take on the value \(-1\) for \( n \) odd and \( x = 0 \). The higher figure on the right shows the graphs of \( s_n \) for \( n = 1, 2, 3, 15, \) and 40.

This month’s phase portrait shows the complex partial sum \( s_5(z) \) in the square \(|\text{Re} z| < 2, |\text{Im} z| < 2 \). For \( z \) complex the series converges in the double sector \(|\text{Im} z| < |\text{Re} z| \); there is no analytic continuation at any of the boundary points, as evidenced by the picture on the right.

Andrey Nikolaevich Kolmogorov (1903 – 1987)

was raised near Yaroslavl by his mother’s sisters, because his unmarried mother died in childbirth when he was born. Later they moved to Moscow, where he finished school. He worked as train conductor and starting in 1920 he attended the university of Moscow.

In 1923, as a pupil of Nikolai Luzin, Kolmogorov constructed an integrable function with a Fourier series that diverges everywhere. This result was internationally recognized because it raised doubts of the correctness of Luzin’s conjecture that the Fourier series of square integrable functions converge pointwise almost everywhere. (The conjecture was proven in 1966 by L. Carleson.)

Kolmogorov was one of the world’s most eminent and versatile mathematicians. He initiated and contributed to many of the important developments of mathematics in the 20th century. Among them are the axiomatization of the theory of probability, the justification of the algorithmic theory of complexity and the theory of cohomology, and the Kolmogorov-Arnold-Moser theory of dynamical systems. In about 500 papers, he made important contributions to logic, topology, Fourier analysis, classical mechanics, theory of turbulent flow, and celestial mechanics. Among the terms that are associated with his name are Kolmogorov axioms, Kolmogorov complexity, Kolmogorov space, Kolmogorov inequality, and Kolmogorov-Sinai entropy.

Kolmogorov was awarded membership to more than 10 learned societies. He received many honorary degrees and many national and international honors. Aside from his work as a researcher, Kolmogorov served as the head of several departments and, from 1964 to 1973, he was president of the Moscow Mathematical Society.
Preholomorphic Functions

Most of the complex functions that are presented in this calendar belong to the class of complex differentiable (holomorphic) functions. This means that in each point \( z \) of its domain \( D \), the limit of the difference quotient exists:

\[
f'(z) = \lim_{h \to 0} \frac{1}{h} \left( f(z + h) - f(z) \right).
\]

The value of the limit (1) will not change if the (complex) increment \( h \) is replaced by \( ih \) or even by \( h + ih \). Thus, for \( h \to 0 \) we have the asymptotic representations

\[
\begin{align*}
f(z + h) &= f(z) + f'(z) h + o(h) \\
f(z + ih) &= f(z) + f'(z) ih + o(h) \\
f(z + h + ih) &= f(z) + f'(z) (h + ih) + o(h).
\end{align*}
\]

If we multiply (2) by \( i \), (3) by \(-i\), and (4) by \(-1\) and add the new equations, then \( f'(z) \) is eliminated and we get

\[
i f(z + h) - i f(z + ih) - f(z + h + ih) + f(z) = o(h).
\]

This equation is an asymptotic relation between the values of a holomorphic function \( f \) at the four vertices \( z, z + h, z + ih, \) and \( z + h + ih \) of a (small) square.

There are different ways to discretize a holomorphic function. For example, one may consider functions that are defined only on a discrete lattice \( D_h \) that consists of all the points in the plane with real and imaginary part a multiple of a positive lattice constant \( h \). The differential quotient (1) for \( h \to 0 \) will no longer make sense, but the validity of (5) with \( o(h) = 0 \) can be used as a substitute for the existence of the limit; that is

\[
f(z + h + ih) - f(z) = i \left( f(z + h) - f(z + ih) \right).
\]

In the 1940s, Jacqueline Ferrand coined the term preholomorphic for complex-valued functions defined on a lattice that satisfy this equation on all points of the lattice. She researched properties of these functions and proved that under suitable conditions the functions will converge to holomorphic functions, if the lattice is refined. With these tools, she was able to give a new proof of the Riemann mapping theorem.

The picture of the month shows the phase portrait of a preholomorphic function on a square lattice with \( 50 \times 50 \) points in the region \(|\text{Re} z| \leq 1.5\) and \(|\text{Im} z| \leq 1.5\). It is adapted from the holomorphic function \( f(z) = e^{4i\pi/3} (\cos(z^4) - 1.1) \). The discrete function was extended constantly from the lattice points to small squares.

Jacqueline Ferrand (1918 – 2014)

was born in Alès in southern France. After attending the lycée in Nîmes, she was one of the first women to be admitted to the École Normale Supérieure in Paris. At the final exam in mathematics, she tied for first place with Roger Apéry. Under the guidance of Arnaud Denjoy (see May), Ferrand started research in mathematics and, in 1942, was promoted with an award-winning thesis on the behavior of conformal mappings on the boundary of the domain.

Ferrand started her academic career in 1943 with an assistant professorship in Bordeaux. Two years later she moved to Caen and, in 1948, she received a call for a professorship to Lille. From 1956 to her retirement in 1984, she held a chair of mathematics at the Sorbonne in Paris.

Ferrand’s main area of work is in geometric transformations (conformal and quasi-conformal mappings of Riemannian manifolds). An invariant that she introduced in 1974 allowed her to solve a major problem in Riemannian geometry.

As a professor and a mother of four children, Ferrand handled an enormous amount of work. In addition to her research, she was also very engaged in teaching. Her lectures were turned into text books that are still in use in France today.
Phragmén-Lindelöf Principle

One of the important tools in complex analysis is the maximum modulus principle, which says that a non-constant function that is holomorphic in a bounded region of the complex plane and continuous on its boundary takes on the maximum of its modulus only on the boundary. This principle, however, does not give any information about functions on unbounded regions. In 1904, the Swedish mathematician Edvard Phragmén succeeded in (partially) extending this principle to unbounded domains. The final extension, however, was achieved four years later in a joint paper with the Finnish mathematician Ernst Leonard Lindelöf. They proved what is now called the Phragmén-Lindelöf principle.

This is a method in which the function in question is multiplied with a specific other function, then the maximum modulus principle on a bounded subdomain is used and, finally, the boundedness on the whole domain using a limit process is established.

A consequence of the Phragmén-Lindelöf principle is that the maximum modulus principle can be extended to unbounded regions provided that the function satisfies a mild growth condition and replacing the maximum by the supremum of the modulus of the function values on the boundary. To illustrate this, we consider the two functions

\[ f(z) = e^{e^{-iz}} \quad \text{and} \quad g(z) = e^{5z} \]

on the unbounded strip \(|\text{Re} z| \leq \pi/2\). On the boundary of this strip, both functions are bounded; \(f\) by 1 and \(g\) by \(e^{5\pi/2}\). However, the function \(f\) grows rapidly and does not satisfy the conditions for the Phragmén-Lindelöf principle. The function \(g\), on the other hand, does satisfy it, and so the principle applies. This yields the conclusion that \(|g(z)| < e^{5\pi/2}\) for all \(z, |\text{Re} z| < \pi/2\). The inset pictures show the two functions for part of the strip (in the window \(|\text{Re} z| \leq 3\pi/4\) and \(2 - 3\pi/4 \leq \text{Im} z \leq 2 + 3\pi/4\)). The portraits are such that the brighter the color the larger the modulus of the function value. In the picture on the left, which is the phase portrait of \(f\), we see that \(f\) grows rapidly in modulus along the positive imaginary axis. For the function \(g\), which is pictured on the right, this is not the case.

This month’s title picture is the function \(f\) in the same window as the one in the inset but using our standard coloration for the phase portrait.

Lars Edvard Phragmén (1863 – 1937)

was born the son of a mathematics teacher in Örebro, Sweden. He started studying mathematics in Uppsala, but soon changed to Stockholm. Mittag-Leffler recognized Phragmén’s mathematical talent quickly and asked him to work for his journal Acta Mathematica. Still a student and as a proofreader for this journal Phragmén discovered major mistakes in a prize winning article by Henri Poincaré. This forced Poincaré to make major revisions extending the paper by 100 pages and pioneering chaos theory – thanking Phragmén for finding the mistakes. Phragmén earned his Ph.D. from the University of Uppsala in 1889.

In 1893, Phragmén was appointed professor at Stockholm University, succeeding Sofia Kovalevskaya who died two years earlier. In 1903, he resigned his post to work for the Royal Inspection of Insurance Companies and, later, to become the director of a private insurance company. But he kept his position as an editor of Acta Mathematica until his death. Phragmén’s main mathematical work was in complex analysis, but he also made significant contributions to insurance mathematics and on voting systems. He was a member of many learned societies, the president of the Swedish Society of Actuaries, and president of the council of the Mittag-Leffler Institute.

<table>
<thead>
<tr>
<th>Su</th>
<th>Mo</th>
<th>Tu</th>
<th>We</th>
<th>Th</th>
<th>Fr</th>
<th>Sa</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>2</td>
<td>3</td>
<td>4</td>
<td>5</td>
<td>6</td>
<td>7</td>
</tr>
<tr>
<td>8</td>
<td>9</td>
<td>10</td>
<td>11</td>
<td>12</td>
<td>13</td>
<td>14</td>
</tr>
<tr>
<td>15</td>
<td>16</td>
<td>17</td>
<td>18</td>
<td>19</td>
<td>20</td>
<td>21</td>
</tr>
<tr>
<td>22</td>
<td>23</td>
<td>24</td>
<td>25</td>
<td>26</td>
<td>27</td>
<td>28</td>
</tr>
<tr>
<td>29</td>
<td>30</td>
<td>31</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
The Liouville Function

Every integer \( n > 1 \) is the product of primes. This factorization is unique up to the order of the primes. As a consequence, the number of prime factors, \( \Omega(n) \), is a well-defined function of \( n \) if the primes are counted with their multiplicity. For instance, since \( 360 = 2^3 \cdot 3^2 \cdot 5 \) we get \( \Omega(360) = 3 + 2 + 1 = 6 \). To extend the function to all positive integers, we define \( \Omega(1) = 0 \). In general, if \( n = p_1^{\alpha_1} \cdots p_k^{\alpha_k} \), then \( \Omega(n) = \alpha_1 + \cdots + \alpha_k \) (with \( \Omega(1) = 0 \)). The Liouville function is defined by

\[
\lambda(n) := (-1)^{\Omega(n)} \quad \text{for} \quad n \in \mathbb{Z}^+.
\]

Since the number of prime factors of a product of two numbers is the sum of the number of prime factors of each of the two numbers, we get \( \lambda(mn) = \lambda(m)\lambda(n) \) for all \( m, n \in \mathbb{Z}^+ \). We now want to calculate the sum \( \sum_{d|n} \lambda(d) \) which ranges over all positive factors \( d \) of the positive integer \( n \).

\[
\sum_{d|n} \lambda(d) = \sum_{\beta_1=0}^1 \cdots \sum_{\beta_k=0}^{a_k} \lambda \left( p_1^{\beta_1} \cdots p_k^{\beta_k} \right) = \sum_{\beta_1=0}^1 (-1)^{\beta_1} \cdots \sum_{\beta_k=0}^{a_k} (-1)^{\beta_k} = \prod_{i=1}^k \frac{1 + (-1)^{\beta_i}}{2}
\]

The domain of \( \lambda \) is a discrete set, making this function inappropriate for a representation by a phase portrait. We turn to its generating function in the form of a Dirichlet series

\[
F(z) := \sum_{n=1}^\infty \frac{\lambda(n)}{n^z}.
\]

Since \( \sum_{n=1}^\infty |\lambda(n)/n^z| = \sum_{n=1}^\infty 1/n^{\Re z} \) the series converges absolutely for at least \( \Re z > 1 \) and, in fact, converges uniformly in each half plane \( \Re z \geq 1 + \varepsilon \) for \( \varepsilon > 0 \). Thus, \( F \) is analytic in the half plane \( \Re z > 1 \). We may multiply \( F \) with the series representation of the Riemann zeta function

\[
\zeta(z) = \sum_{n=1}^\infty \frac{1}{n^z}
\]

that is also valid in the same half plane (see Complex Beauties, November 2011) and obtain

\[
F(z)\zeta(z) = \sum_{n=1}^\infty \frac{\lambda(n)}{n^z} \sum_{m=1}^\infty \frac{1}{m^z} = \sum_{n=1}^\infty \sum_{m=1}^\infty \frac{\lambda(n)}{(nm)^z} = \sum_{k=1}^\infty \sum_{n|k} \frac{\lambda(n)}{k^z} = \sum_{m=1}^\infty \frac{1}{m^{2z}} = \zeta(2z).
\]

This results in the representation \( F(z) = \zeta(2z)/\zeta(z) \) which has an analytic continuation to \( \mathbb{C} \) just like the zeta function has. The picture of the month shows the phase portrait of this function.

Joseph Liouville (1809 – 1882)

was born in Saint-Omer (northern France) the son of an officer in Napoleon’s army. He grew up in Toul near Nancy. After studies at the École Polytechnique, he attended the École des Ponts et Chaussées for a short time. Because the work as an engineer did not agree with his health, he started a career in academia with positions at the École Polytechnique and at the Collège de France. He even beat out Cauchy for a position.

Liouville worked in many different fields: complex analysis (Liouville’s theorem, conformal mappings), number theory (first examples of transcendental numbers), analysis (fractional calculus, existence of non elementary integrals), eigenvalue problems in differential equations (Sturm-Liouville theory), mathematical physics and astronomy. He published more than 400 papers. In 1836, he founded Journal de Mathématiques Pures et Appliquées (also sometimes known as the Journal de Liouville), which is still highly regarded today. Among other things, he was the original publisher of works by Évariste Galois in this journal.