Seeing the Monodromy Group of a Blaschke Product

Elias Wegert, TU Bergakademie Freiberg

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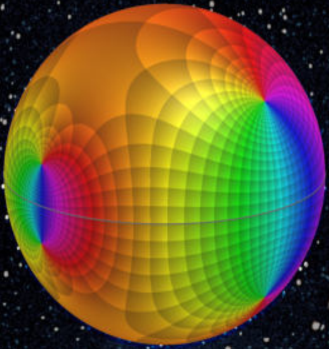

Visualizing Complex Functions

Visualization of complex functions

Graphical representations of functions belong to the most useful tools in mathematics and its applications. However, the graph

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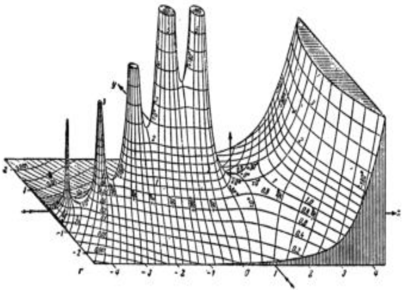
This picture of the complex Gamma function, published 1909 in the famous book by Jahnke and Emde, acquired an almost iconic status.



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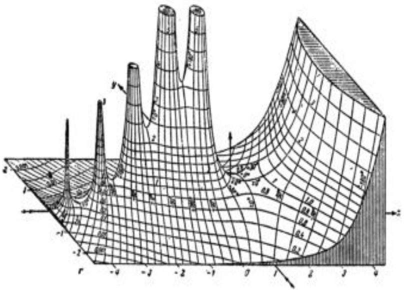
The analytic landscape depicts only the absolute value of a function and neglects its argument (phase). Jahnke and Emde compensated this by drawing lines of constant argument.



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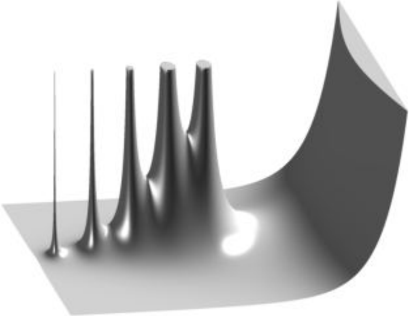

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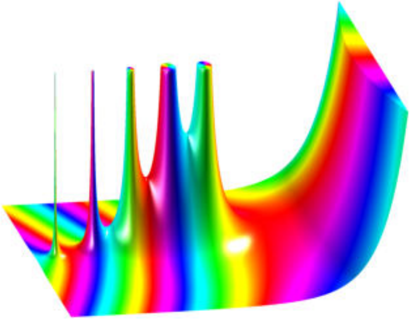
Today analytic landscapes can be computed easily, and coloring allows one also to incorporate the argument.



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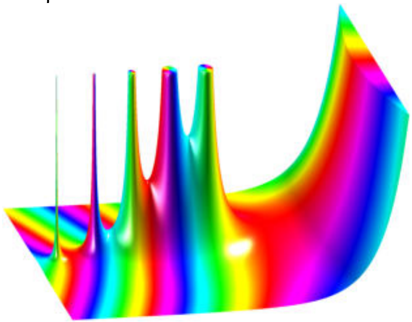
Instead of the argument one better uses the (well-defined) phase

$$f(z)/|f(z)|.$$

It lives on the unit circle $\mathbb{T}$, and can be encoded using a circular color scheme.



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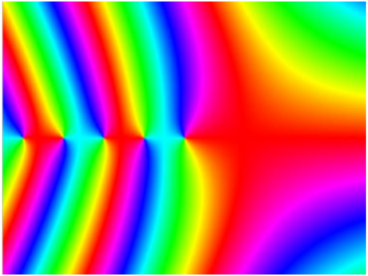
Viewing the colored analytic landscape straight from top, we see (what I call) the phase portrait or phase plot of the function.



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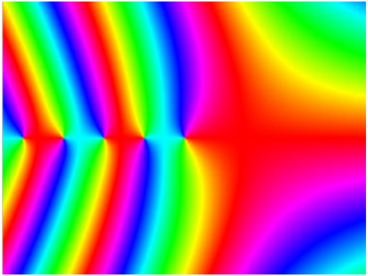
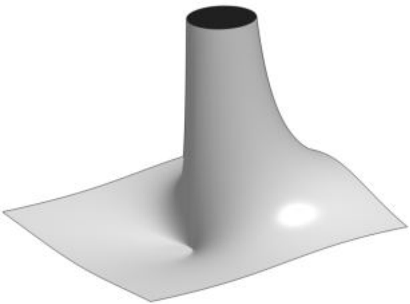
Phase plots are special variants of domain coloring.



Phase plots outperform analytic landscapes

The phase plot of a function shows many properties more clearly than the analytic landscape.



An analytic landscape of $f(z) = \mathrm{e}^{1/z}$

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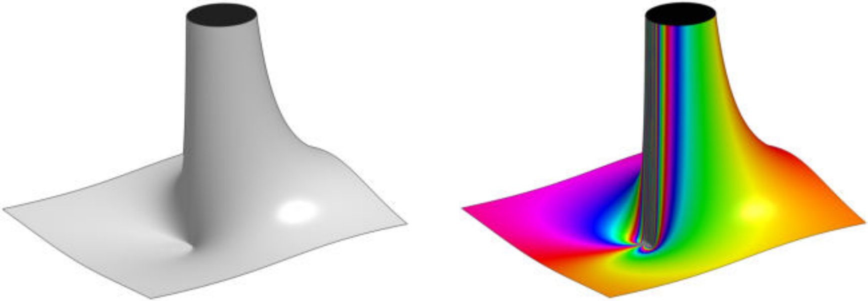

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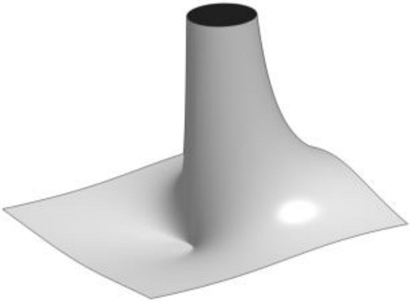

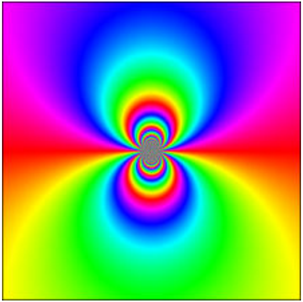

An analytic landscape of $f(z) = \mathrm{e}^{1/z}$ and its phase plot.

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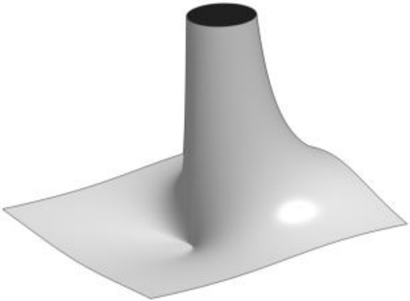
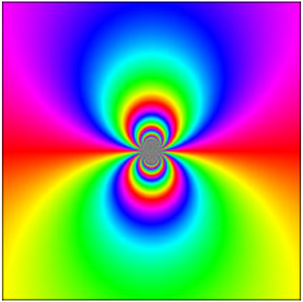




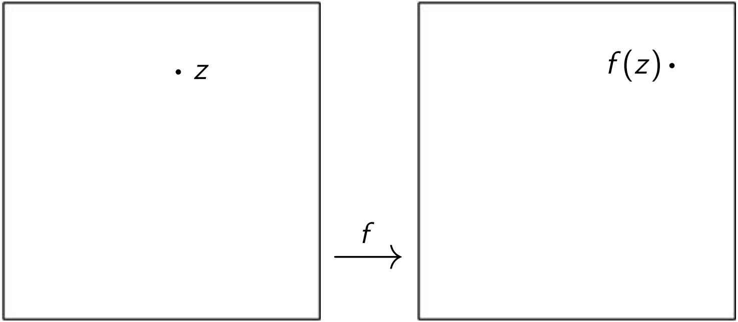
An analytic landscape of $f(z) = \mathrm{e}^{1/z}$ and its phase plot.
A function which is meromorphic in an open connected set (domain) G is uniquely determined up to a positive constant factor by its phase plot.

Phase Plots: Less is more

Phase Plots: Less is more – more or less

Phase plots and their modifications

We illustrate the construction of a phase plot with the rational function $f(z) = (z-1)/(z^2+z+1)$ in the square $|\mathrm{Re}\, z| \le 2$, $|\mathrm{Im}\, z| \le 2$.



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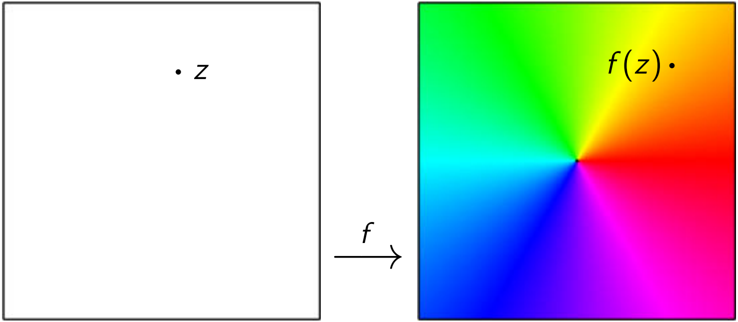

All points of the w-plane with the same argument get the same color.

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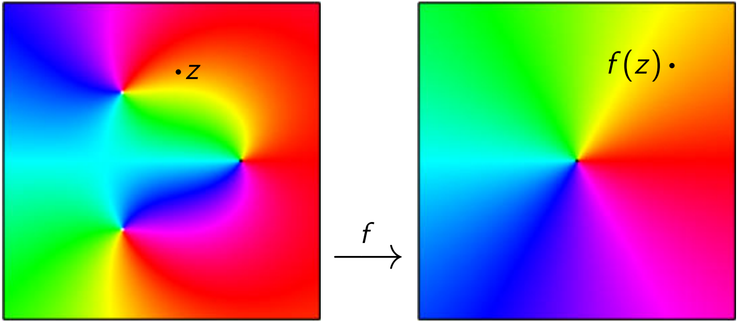

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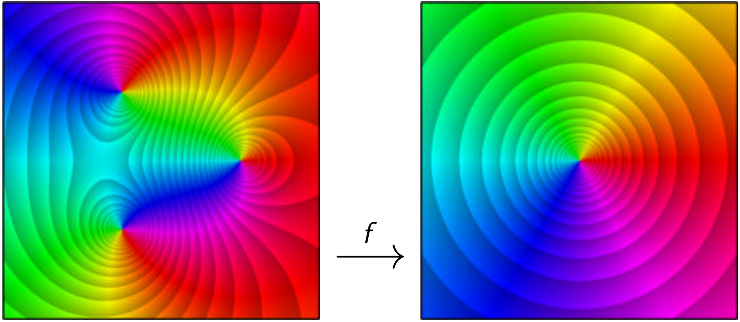

Modifications of the color scheme in the w-plane allow one to read off properties of the function more easily.
This version incorporates the absolute value of f by highlighting some contour lines of $|f|$.

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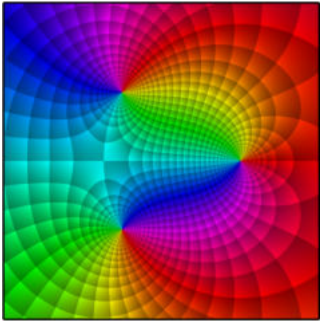
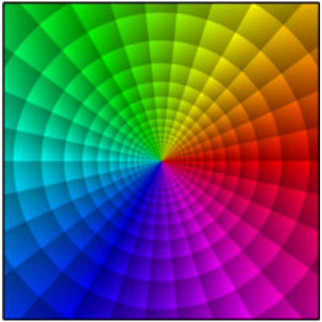


$\xrightarrow{f}$



Modifications of the color scheme in the w-plane allow one to read off properties of the function more easily.
This variant demonstrates that the mapping f is conformal.
With a few exceptions, all "tiles" have four right angled corners.

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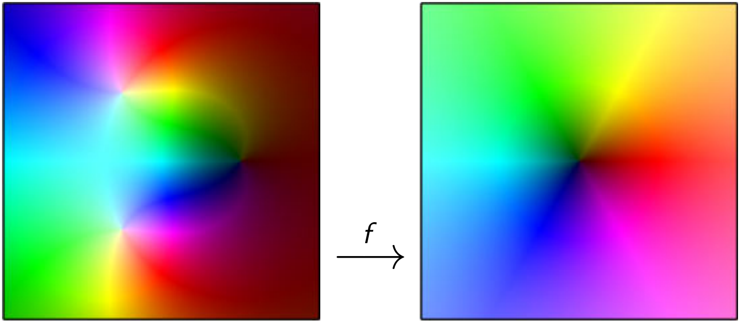
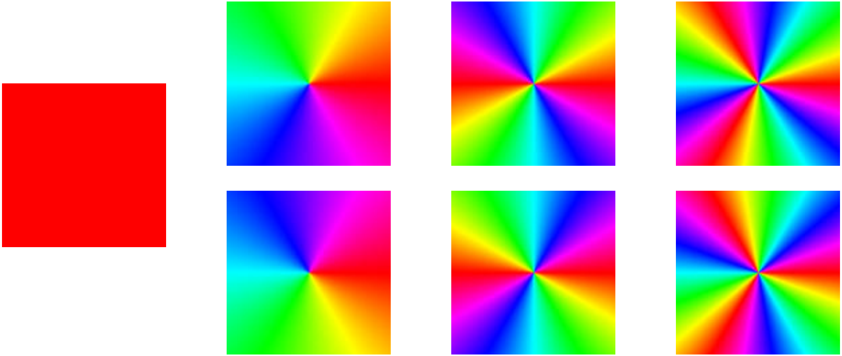


Classical domain coloring (Frank Farris) uses a two-dimensional color scheme, with brightness corresponding to absolute value, to encode the values of f completely.

How to read it

Zeros and poles

Both rows show phase portraits of the power functions $f(z) = z^k$ for $k = 0$ (left), $k = 1, 2, 3$ (above) and $k = -1, -2, -3$ (below).



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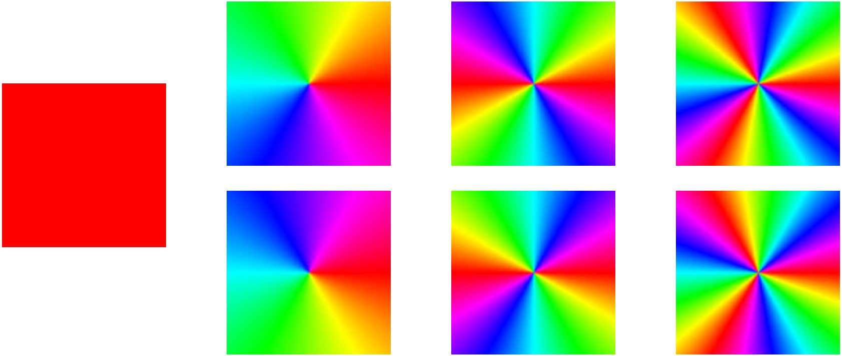
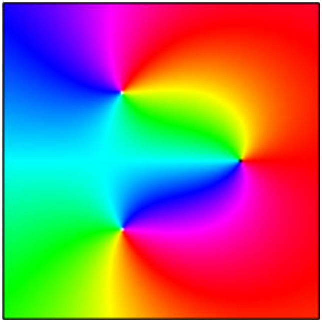


Zeros and poles can be distinguished by the orientation of colors, their multiplicity can be read off easily.

Isochromatic lines and contour lines

The isochromatic lines (sets with equal phase of f)



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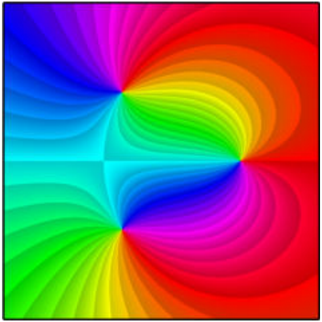

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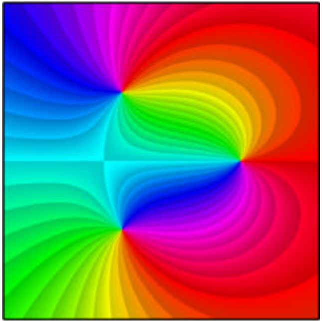
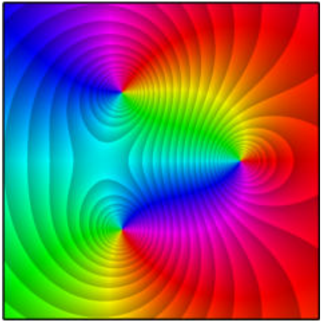
The isochromatic lines (sets with equal phase of f) and the contour lines (sets with equal absolute value of f)





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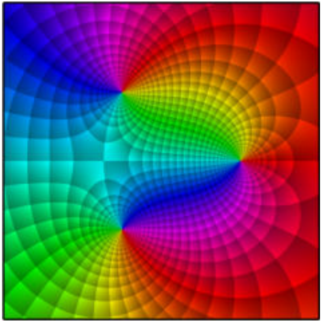
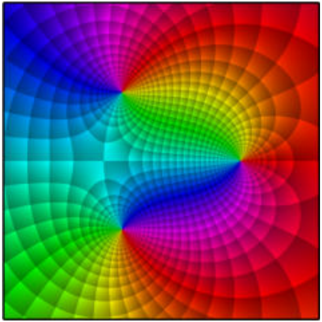
The isochromatic lines (sets with equal phase of f) and the contour lines (sets with equal absolute value of f) are perpendicular.





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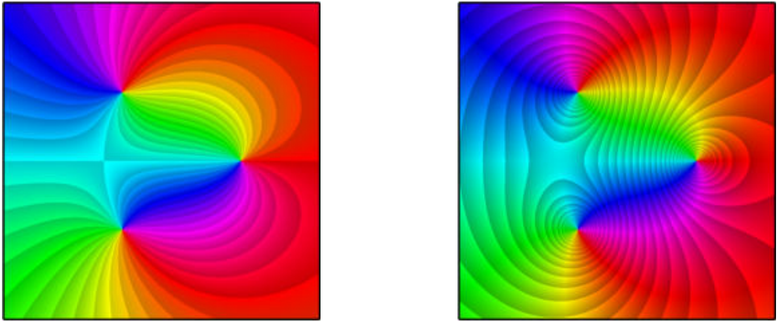

The density of these lines is related to the relative growth of the function, it is proportional to $|f'|/|f| = |(\log f)'|$.

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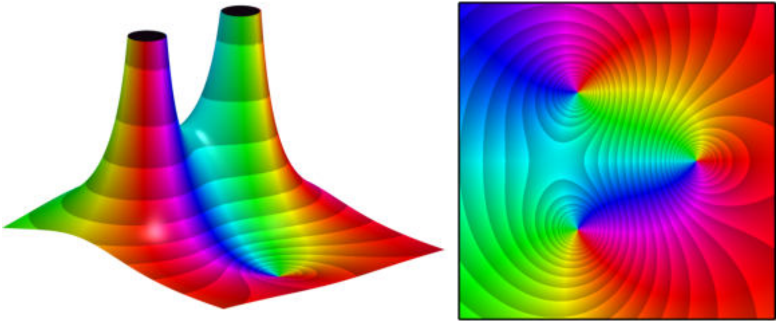

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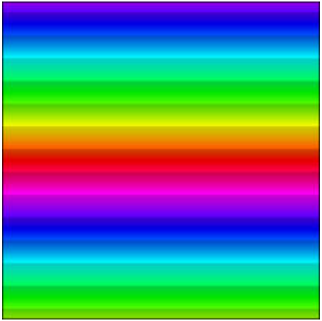

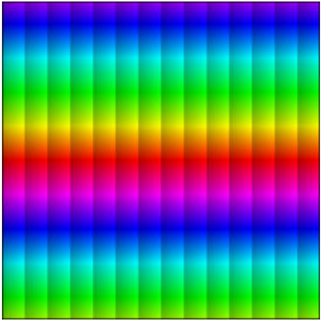

For the exponential function both families consist of parallel lines.
Here we see $f(z) = \exp(z)$ in $|\mathrm{Re}\, z| < 5$, $|\mathrm{Im}\, z| < 5$.

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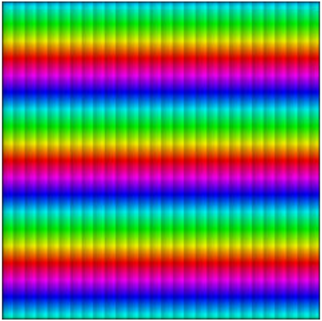

For the exponential function both families consist of parallel lines.
Here we see $f(z) = \exp(2z)$ in $|\mathrm{Re}\, z| < 5$, $|\mathrm{Im}\, z| < 5$.

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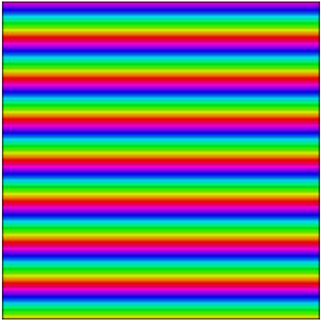
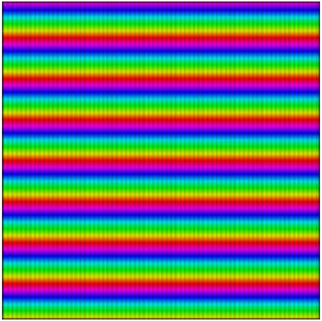
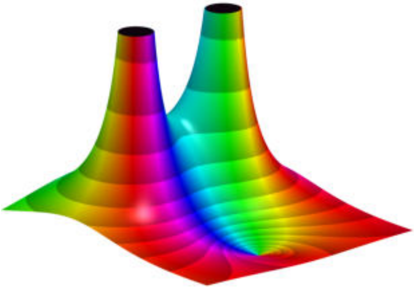


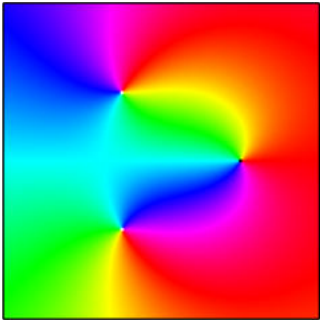


For the exponential function both families consist of parallel lines.
Here we see $f(z) = \exp(5z)$ in $|\mathrm{Re}\, z| < 5$, $|\mathrm{Im}\, z| < 5$.

Critical Points and Saddle Points

Critical points ζ of a function f are the zeros of its derivative.
Points where $f'(\zeta) = 0$ and $f(\zeta) \neq 0$ are called saddle points.





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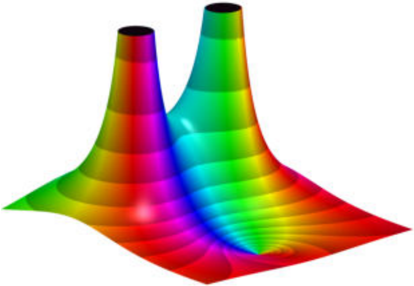
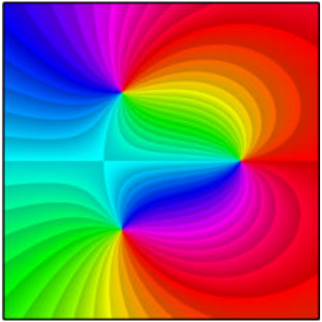
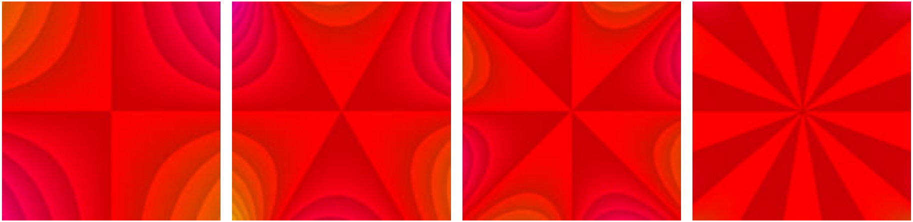




In the phase plot of f saddle points are the only crossing points of isochromatic lines.

The Order of Saddle Points

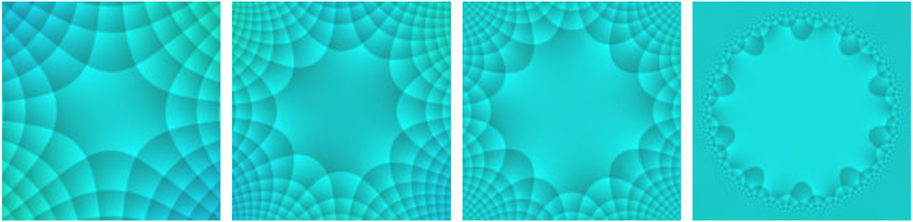
The *order* of a saddle point is the multiplicity of the zero of f'.
A saddle point of order n is the crossing of $n+1$ isochromatic lines.



The saddle points in these phase plots have orders 1,2,3 and 8.

The Order of Saddle Points

A tile containing saddle points is called exceptional. When it has saddle points of orders summing up to n, it has $4(n+1)$ corners.



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Blaschke Products

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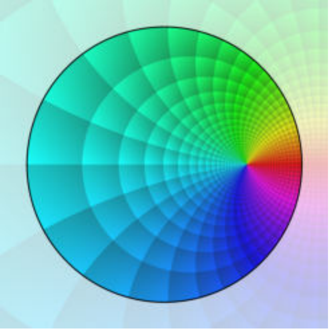
A Blaschke factor is a Moebius transformation of the form

$$f(z) = c\,\frac{z - z_0}{1 - \overline{z_0} z}, \qquad |z_0| < 1,\ |c| = 1.$$



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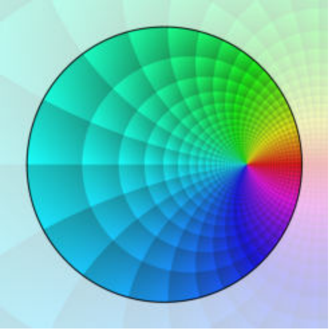
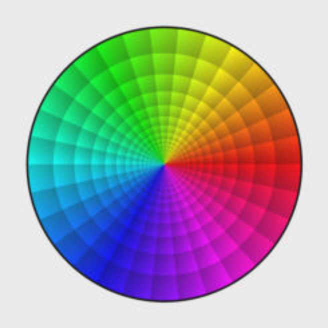
The mapping $f : \mathbb{D} \to \mathbb{D}$ is a conformal automorphism of the unit disk $\mathbb{D}$.





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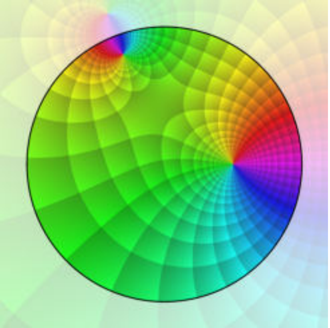


A Blaschke product is the product of Blaschke factors,

$$B(z) = c \prod_{k=1}^{n} \frac{z - z_k}{1 - \overline{z_k}z},$$

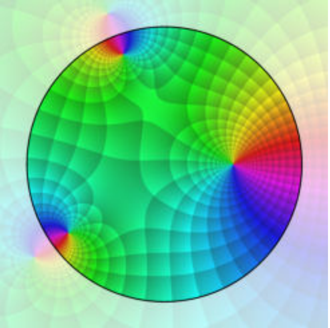
it has modulus 1 on $\mathbb{T}$.
This has degree two.

Blaschke Products

A Blaschke factor is a Moebius transformation of the form

$$f(z) = c\,\frac{z - z_0}{1 - \overline{z_0}z}, \qquad |z_0| < 1,\ |c| = 1.$$

The mapping $f : \mathbb{D} \to \mathbb{D}$ is a conformal automorphism of the unit disk $\mathbb{D}$.



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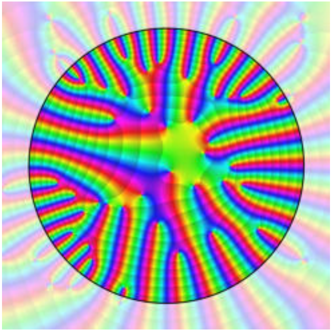

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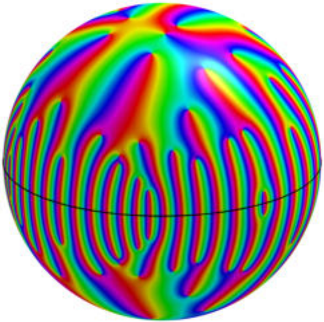

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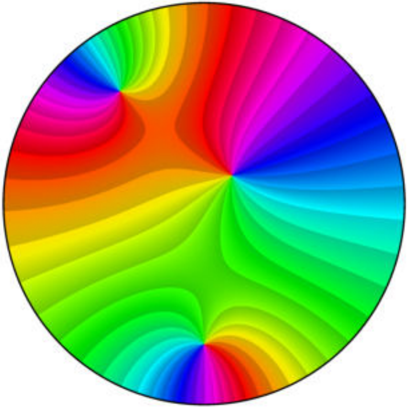
Due to $B(1/\overline{z}) = 1/\overline{B(z)}$, the phase plot of Blaschke products on the Riemann sphere is symmetric with respect to the equator.

Intermezzo: The Phase Flow

The phase flow

The phase plot of Blaschke products suggests that “phase” is a substance which flows along the isochromatic lines.



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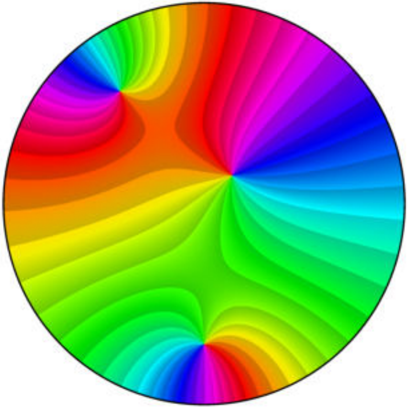

This can be modeled by a vector field. If $f : D \to \widehat{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$ is a meromorphic function, then V_f defined by

$$V_f(z) := -\frac{f(z)\,\overline{f'(z)}}{|f(z)|^2 + |f'(z)|^2}$$

is smooth on D, and $V_f(z)$ is tangent to the isochromatic lines of f at z (with $\mathbb{C} \cong \mathbb{R}^2$).

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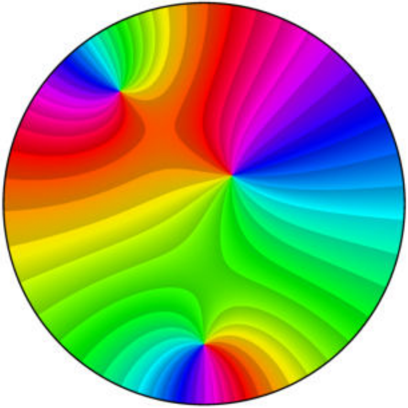

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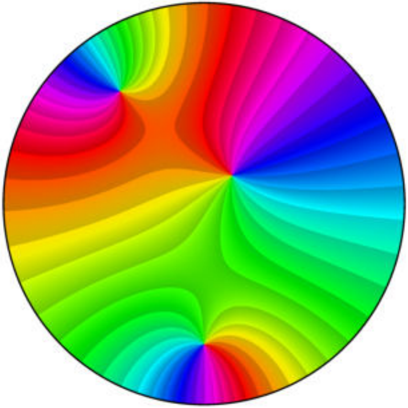
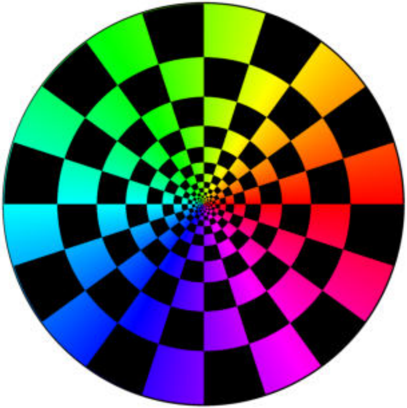
The vector field V_f generates a continuous semigroup, the phase flow Ψ_f.

Visualization of the phase flow

Visualization of the proper phase flow is demanding, here is a cheap substitute. It has the same orbits but a different (discontinuous) speed.

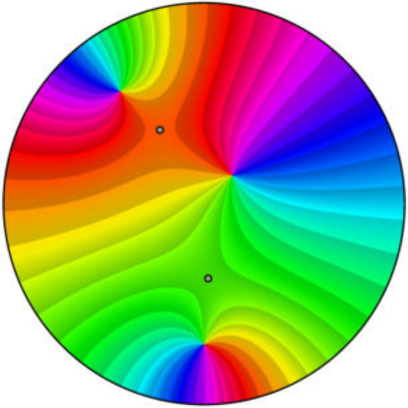




The animated phase plot is a pull-back of the range disk, covered by a rotating polar chessboard mask.

Basins of zeros

The phase flow of a meromorphic function f has fixed points at its zeros (attracting), saddle points (as the name tells) and poles (repelling).



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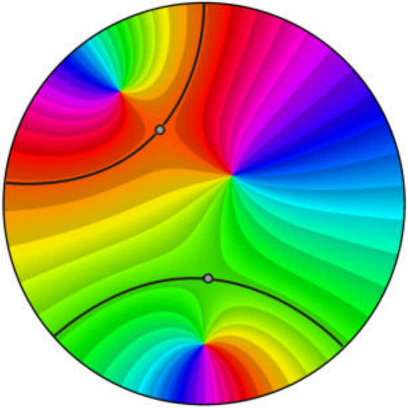

The phase flow of a Blaschke product B is special: removing from $\mathbb{D}$ all stable manifolds of the saddle points, the remaining set $\mathbb{D} \setminus S$ is the disjoint union of simply connected domains, which are the basins of zeros.

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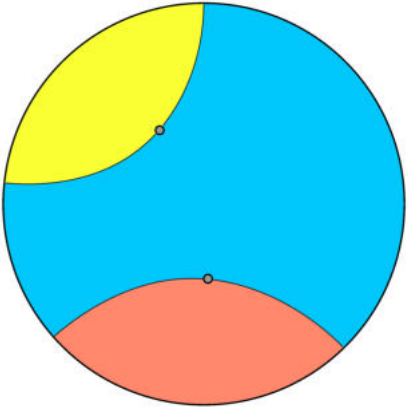

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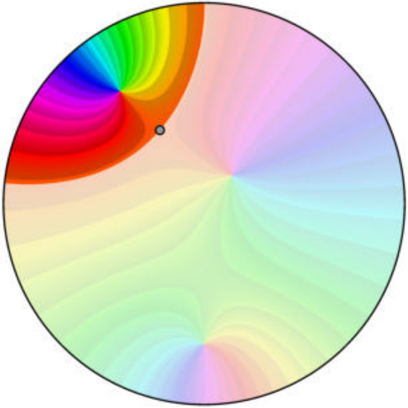

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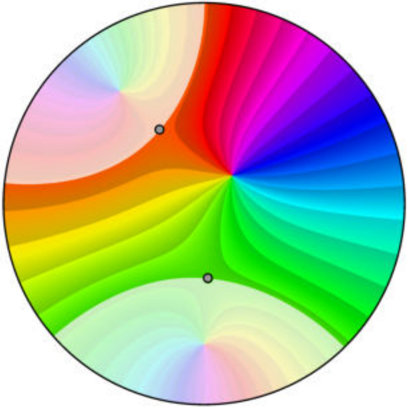

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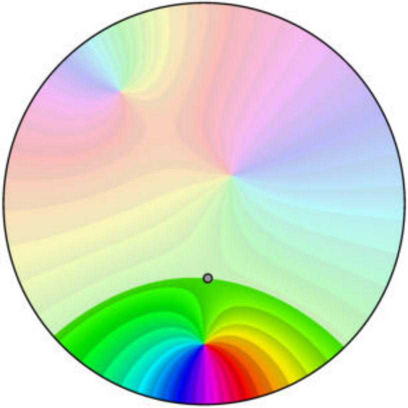

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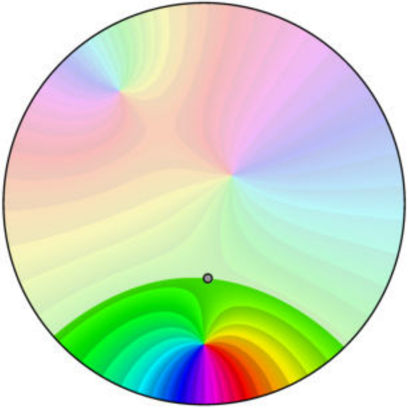

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This statement must be modified somewhat when B has multiple zeros.

Regularized Blaschke products

The basins of attraction of the zeros of B are natural candidates to form the sheets of the Riemann surface of B^{-1}.

For the following constructions we assume that B is regularized, i.e.,

(1) all zeros of B are simple (which implies that 0 is not a critical value),

(2) if ζ_j and ζ_k are critical points of B, then

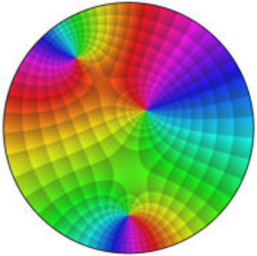
$$\begin{aligned} |B(\zeta_j)| = |B(\zeta_k)| \quad &\Longrightarrow \quad B(\zeta_j) = B(\zeta_k) \\ B(\zeta_j)/B(\zeta_k) \in \mathbb{R}_+ \quad &\Longrightarrow \quad B(\zeta_j) = B(\zeta_k). \end{aligned}$$

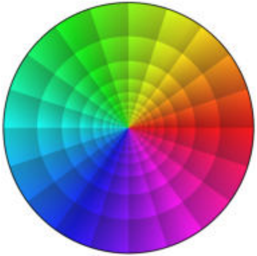
These are formal restrictions – they can always be achieved by replacing B by $\widetilde{B} = B_2 \circ B \circ B_1$, where B_1 and B_2 are appropriate Blaschke products of degree 1 (conformal automorphisms of $\mathbb{D}$). This transformation has no influence on the structures we are interested in.

The Riemann Surface of B^{-1}

Blaschke products as covering maps

A Blaschke product $B : \mathbb{D} \to \mathbb{D}$ of degree n is an n-fold covering map.





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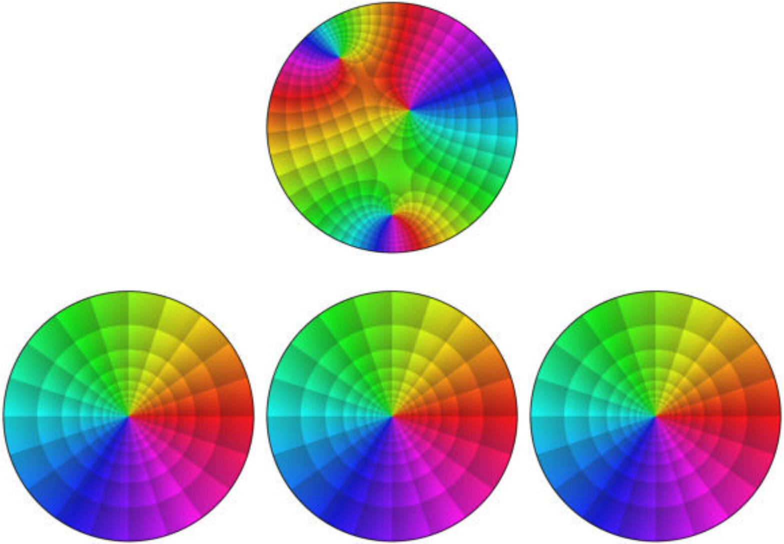
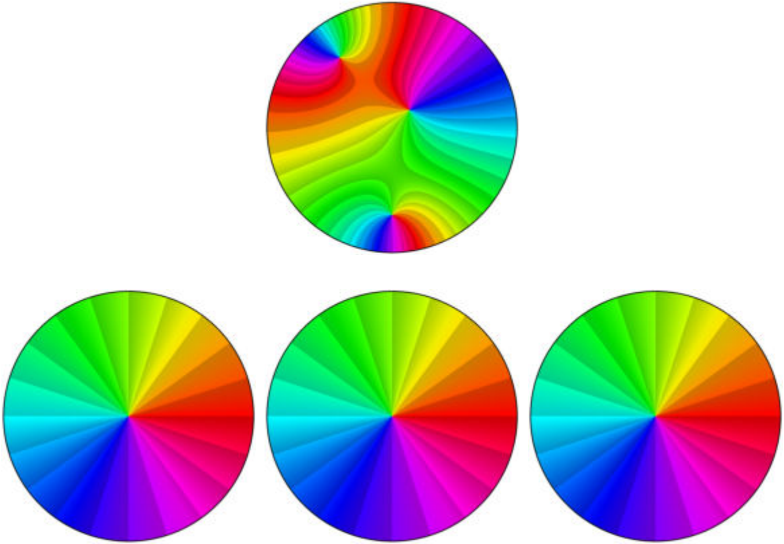


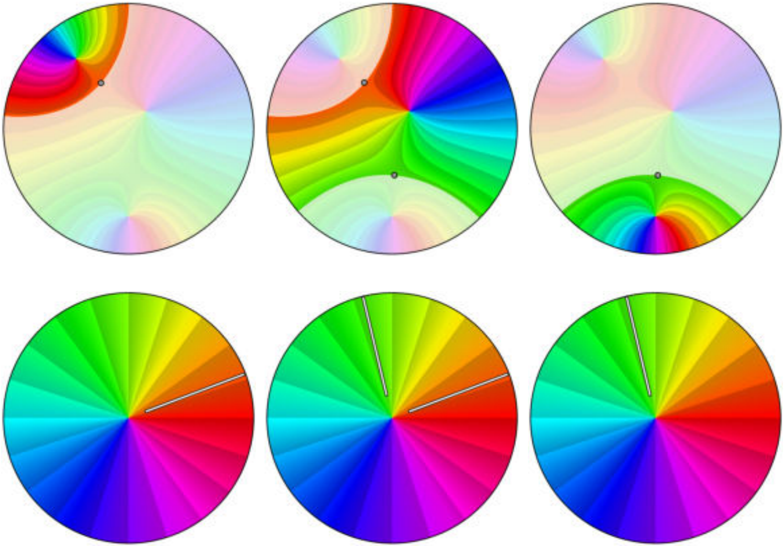
Blaschke products as covering maps

The phase flow allows us to determine the basins of the zeros.



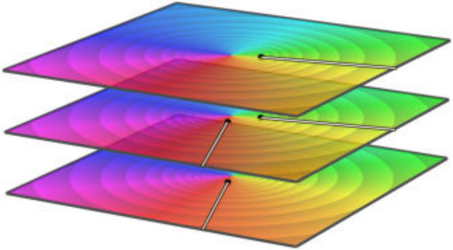
Blaschke products as covering maps

Each basin is mapped onto a slit disk.



Riemann surfaces of Blaschke products

Since the Blaschke product is an n-fold covering map of $\mathbb{D}$ onto itself, its inverse B^{-1} lives on a Riemann surface S_B formed by n sheets $D_1, \ldots, D_n$, where each sheet is a copy of $\mathbb{D}$.



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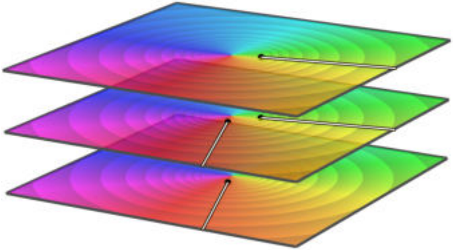

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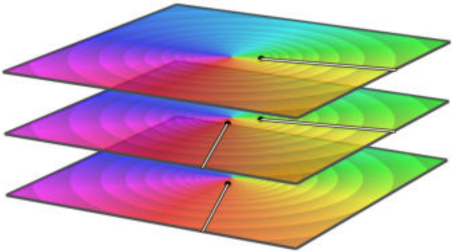

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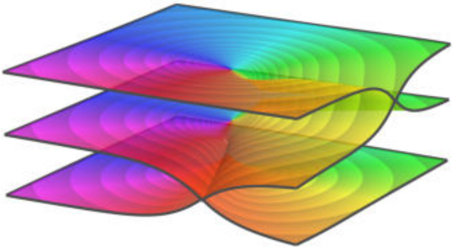

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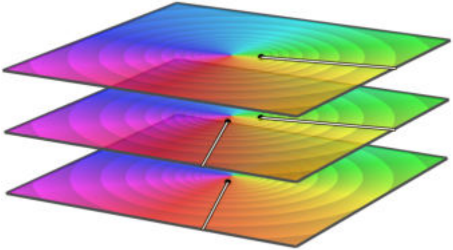

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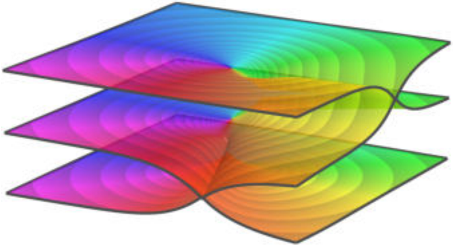

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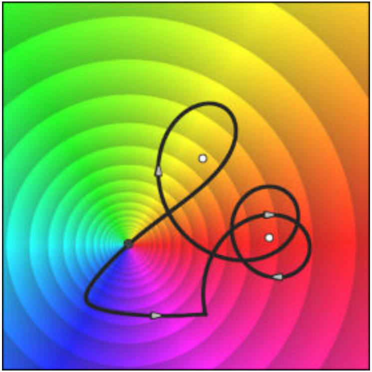
Blaschke Products: Monodromy

The Fundamental Group

Let $W = B(S)$ be the set of critical values of a regularized Blaschke product B, and consider closed oriented paths (loops) γ in $\dot{\mathbb{D}} := \mathbb{D} \setminus W$



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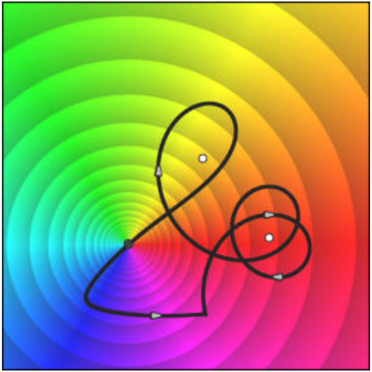


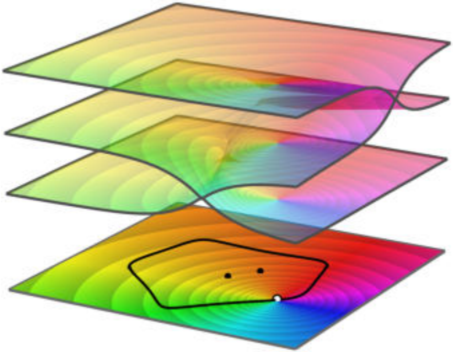
with base point $0 \notin W$.

These loops form a *group* with respect to *concatenation*.

The fundamental group $\pi_1(\dot{\mathbb{D}})$ of $\dot{\mathbb{D}}$ consists of equivalence classes $[\gamma]$ of *homotopic loops* in $\dot{\mathbb{D}}$.

Monodromy Group of a Blaschke Product

Any path γ in $\dot{\mathbb{D}}$ can be *lifted* to a path Γ on the Riemann surface S_B.



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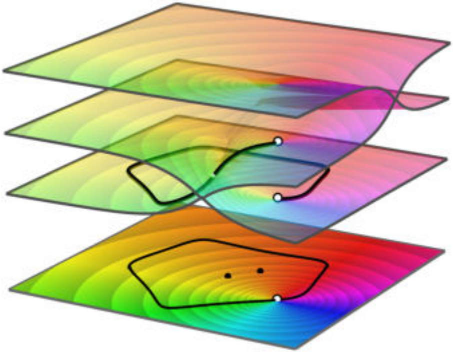

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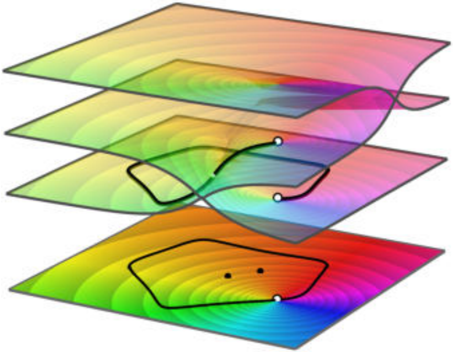

The initial point of Γ can be chosen on any sheet ("above" the initial point of γ); once this point is fixed, Γ is uniquely determined. If γ is a loop, this need not be so for Γ, since the sheet of its terminal point can be different from the sheet of its initial point.

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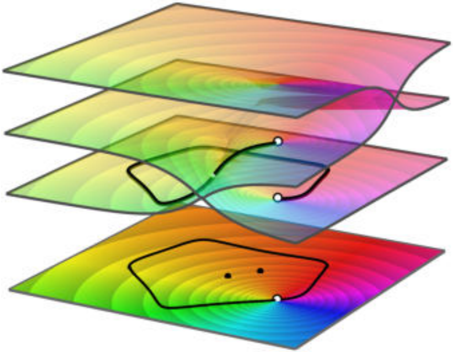

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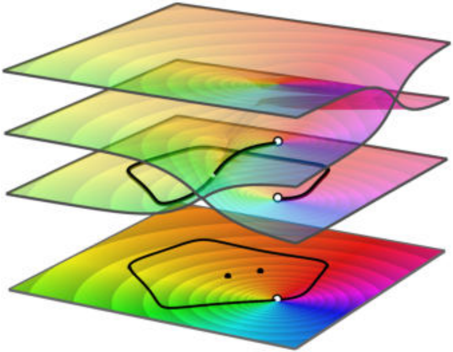

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Since M_γ depends only on the homotopy class of γ, we write $M_{[\gamma]}$.

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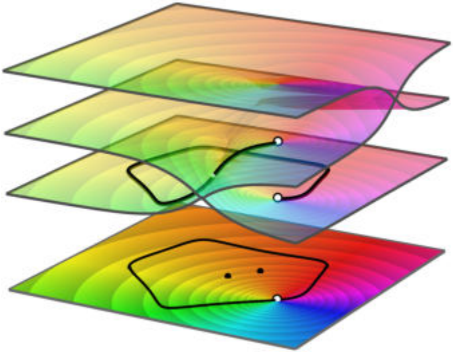

This defines the *monodromy mapping*

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$M_B([\gamma])$ is the permutation of sheets of S_B induced by the lifting of a closed loop γ.
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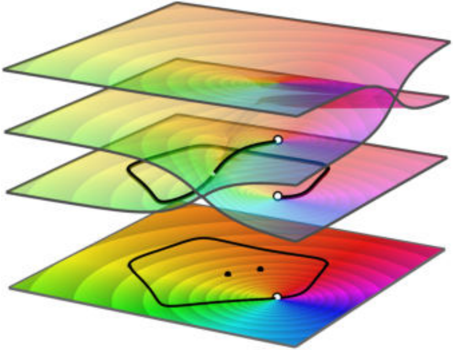

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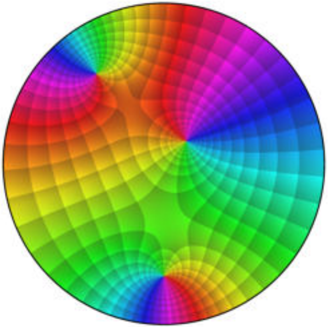
Endowed with concatenation of loops, $M_B : \pi_1(\dot{\mathbb{D}}) \to \mathbb{S}_n$ is the monodromy group of B, a subgroup of the *symmetric group* $\mathbb{S}_n$.

Generators of the monodromy group

The fundamental group $\pi_1(\dot{\mathbb{D}})$ is generated by the (equivalence classes of) "small loops" γ_j around the *critical values* $B(\zeta_j)$, and the monodromy group M_B is generated by the permutations of the sheets $D_1, \ldots, D_n$ induced by these loops. This can be seen in the phase plot of B !



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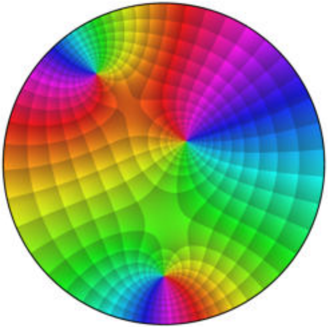

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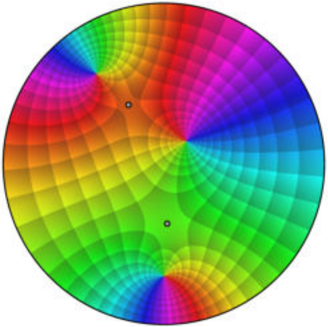

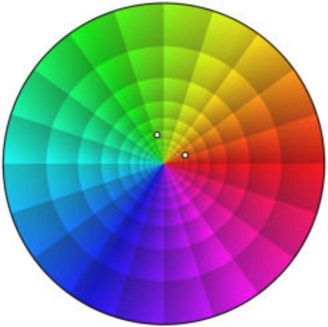

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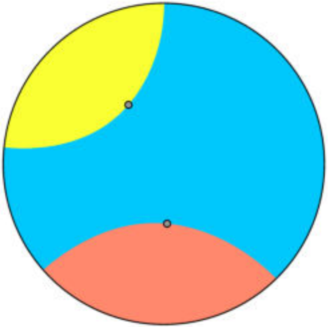

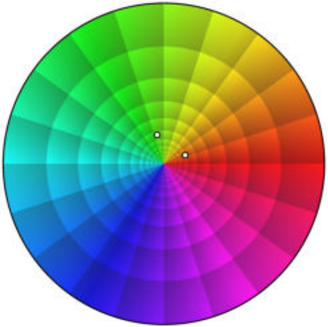

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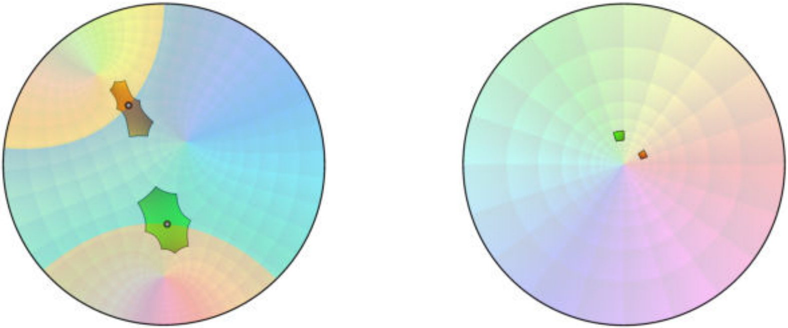

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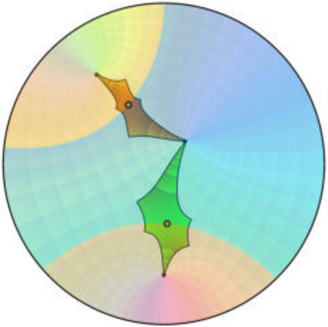

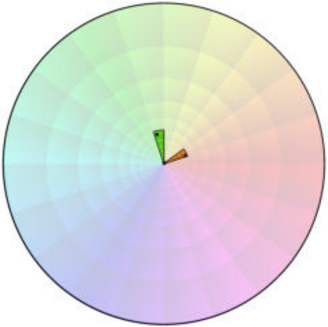

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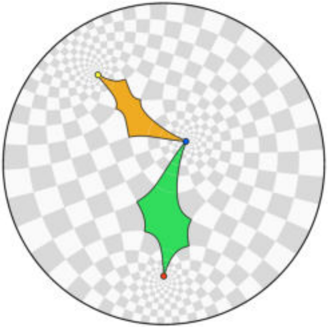

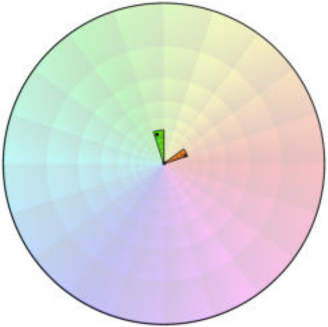

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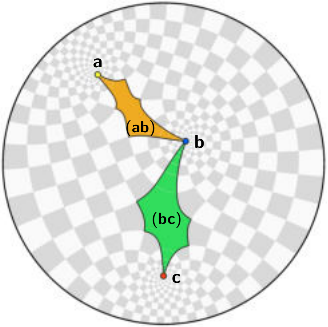

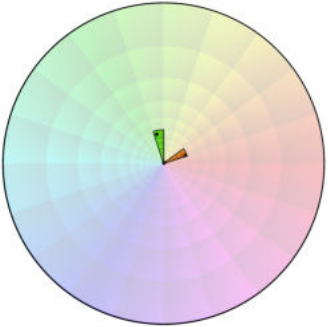

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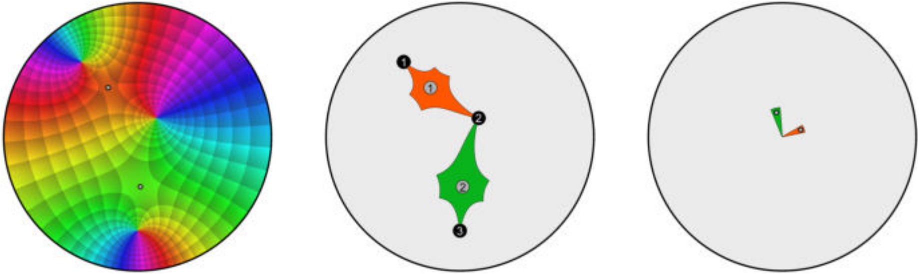
Some examples

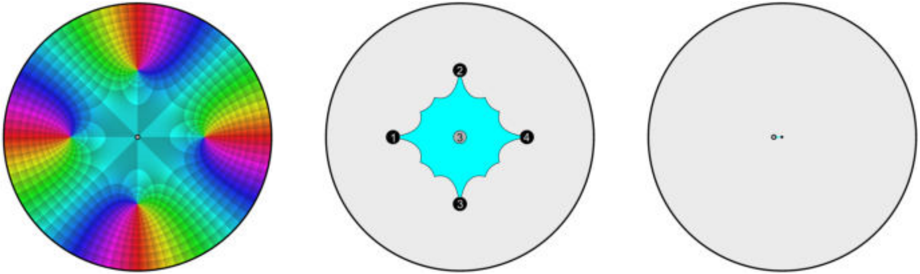
Phase plot of a Blaschke product (left), generators of its monodromy group (middle), and loops from which the cells are generated (right).



This Blaschke product has degree 3, with two saddle points of order 1. The generators of its monodromy group are $(1\,2)$ and $(2\,3)$, and M_B is the symmetric group $\mathbb{S}_3$.

Some examples

Phase plot of a Blaschke product (left), generators of its monodromy group (middle), and loops from which the cells are generated (right).



This Blaschke product has degree 4, with a saddle point of order 3.
The generator of its monodromy group is $(1\,2\,3\,4)$, so that $M_B = \mathbb{Z}_4$.
The (only) critical value is very small

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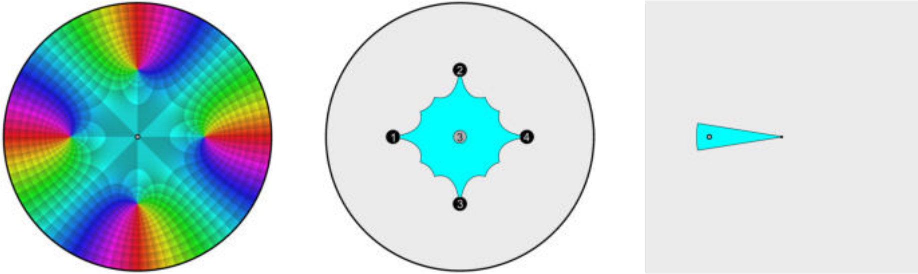
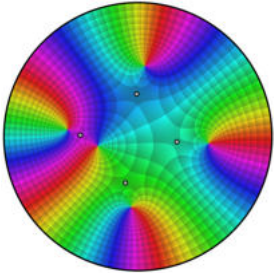
Phase plot of a Blaschke product (left), generators of its monodromy group (middle), and loops from which the cells are generated (right).

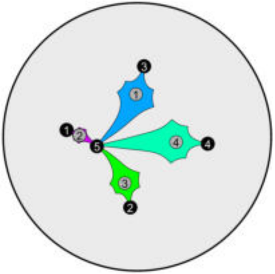


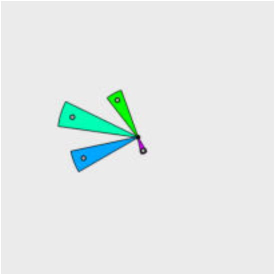
This Blaschke product has degree 4, with a saddle point of order 3. The generator of its monodromy group is $(1\,2\,3\,4)$, so that $M_B = \mathbb{Z}_4$. The (only) critical value is very small , a zoom-in shows the loop more clearly.

Some examples

Phase plot of a Blaschke product (left), generators of its monodromy group (middle), and loops from which the cells are generated (right).



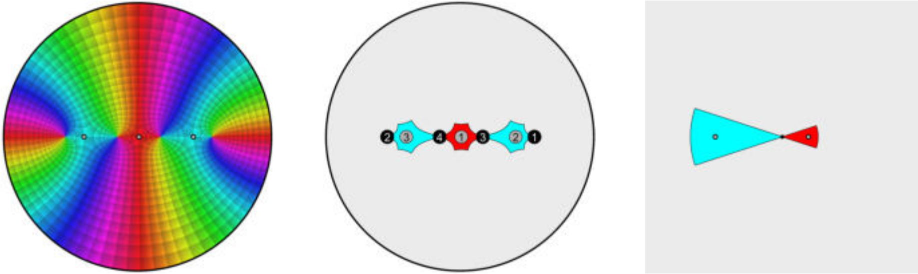




A generic Blaschke product of degree 5 has four saddle points of order 1. The generators of its monodromy group are (1 5), (2 5), (3 5), (4 5), and the monodromy group is the symmetric group $\mathbb{S}_5$.

Some examples

Phase plot of a Blaschke product (left), generators of its monodromy group (middle), and loops from which the cells are generated (right).



This Blaschke product has degree 4 and 3 saddle points $\zeta_1, \zeta_2, \zeta_3$ of order 1, but two critical values coincide, $B(\zeta_2) = B(\zeta_3) =: w$.

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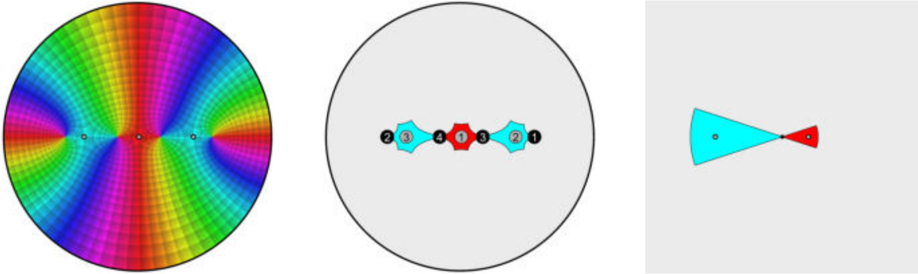

This Blaschke product has degree 4 and 3 saddle points $\zeta_1, \zeta_2, \zeta_3$ of order 1, but two critical values coincide, $B(\zeta_2) = B(\zeta_3) =: w$.
Since a loop which encircles w affects both cells C_2 and C_3, they "act simultaneously", which results in the permutation $(1\,3)(2\,4)$.
Together with the second generator $(3\,4)$ this produces the monodromy group of B, which is the dihedral group $\mathbb{D}_4$.

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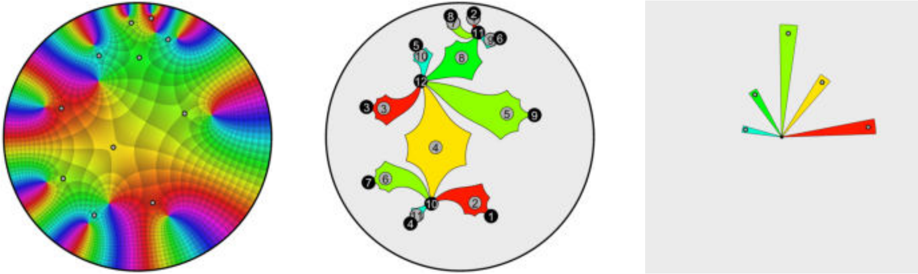
Phase plot of a Blaschke product (left), generators of its monodromy group (middle), and loops from which the cells are generated (right).



Now we have $\deg B = 12$ with 11 saddle points, but only 5 different critical values: $w_1 = w_2 = w_3$, $w_5 = w_6 = w_7$ and $w_9 = w_{10} = w_{11}$.

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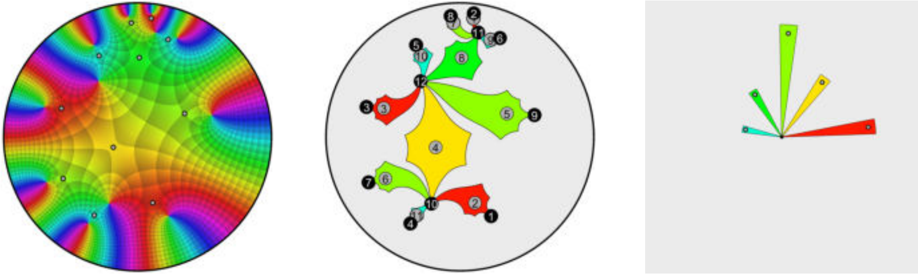

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$(1, 10)(2, 11)(3, 12)$, $(10, 12)$, $(7, 10)(8, 11)(9, 12)$, $(11, 12)$, $(4, 10)(5, 12)(6, 11)$

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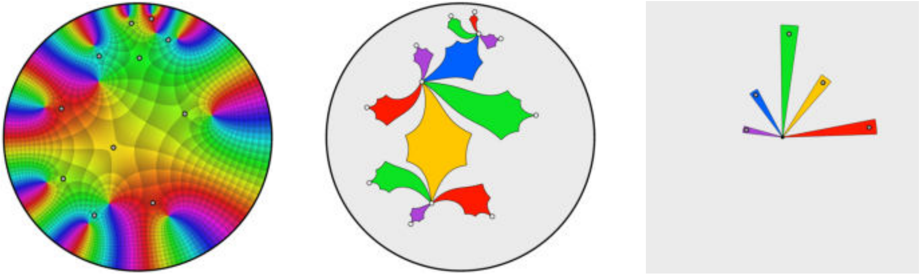

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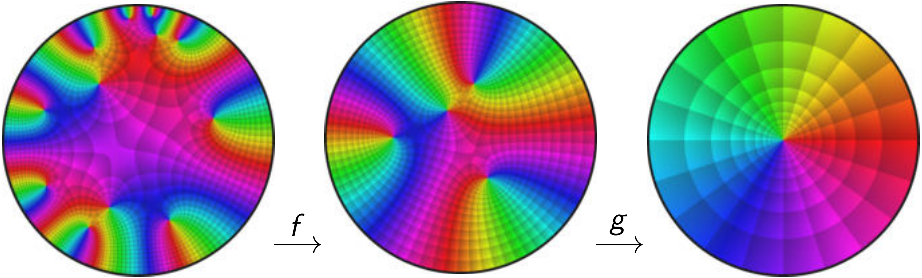
Blaschke Products: Composition

Compositions of Blaschke Products

The Blaschke product B in the last example was special, because it was a composition of two Blaschke products of lower degree, $B = g \circ f$.



The figure illustrates how Blaschke products f of degree 3 and g of degree 4 are composed to a Blaschke product of degree 12.

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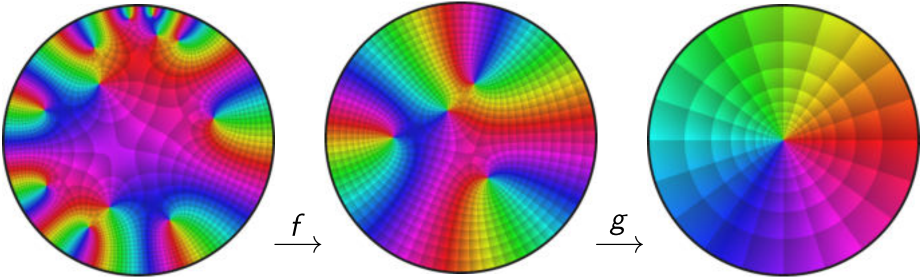

The figure illustrates how Blaschke products f of degree 3 and g of degree 4 are composed to a Blaschke product of degree 12.
Since a phase plot is constructed by pulling back the structure from the range plane to the domain, it should be read from right to left.

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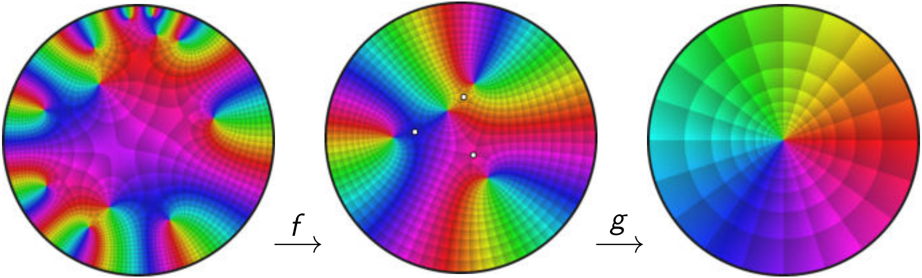

The phase plot of g (middle) shows the critical points of g.

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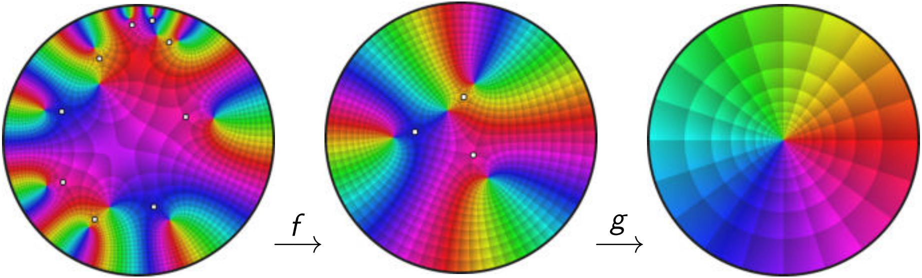

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By the chain rule,

$$(g \circ f)' = (g' \circ f) \cdot f',$$

their *pull back* via f are critical points of B (left).

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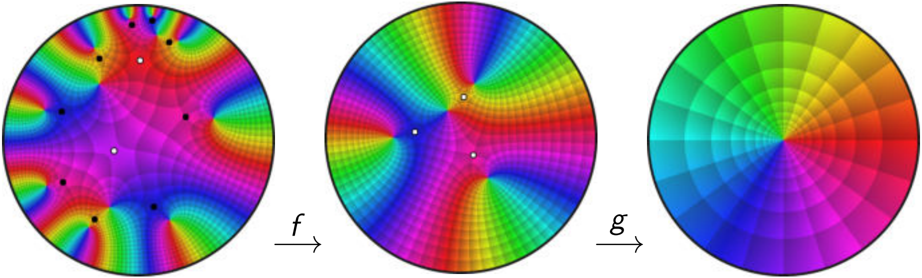

The phase plot of g (middle) shows the critical points of g.
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The remaining critical points of B are the critical points of f.

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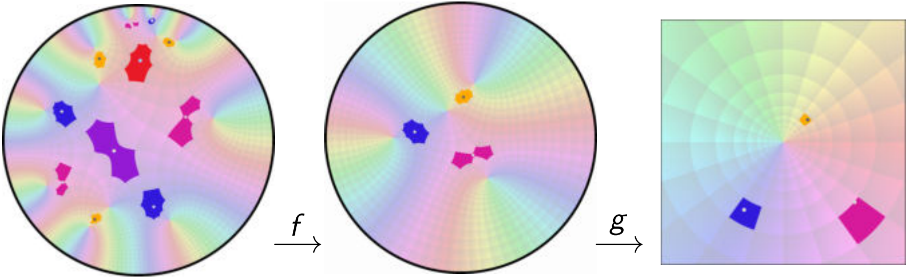

This can also be seen in the corresponding exceptional tiles.

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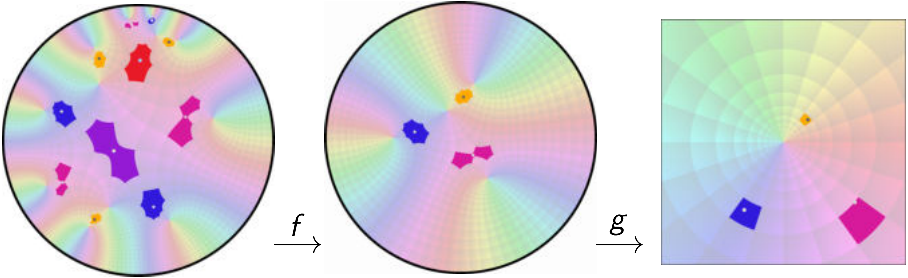

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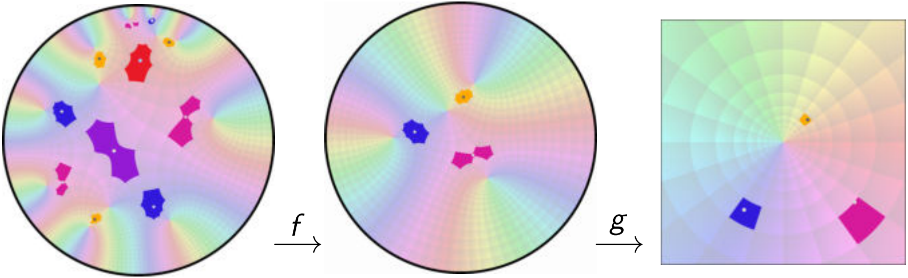


This can also be seen in the corresponding exceptional tiles.
Since f maps $\mathbb{D}$ onto a 3-fold covering of $\mathbb{D}$, each exceptional tile of g is triplicated in the phase plot of B.
All tiles with the same color are conformally equivalent, since they are pulled back from the same tile in the image on the right.

A Criterion for Decomposability

Theorem (Daepp, Gorkin, Shaffer, Sokolowsky, Voss, 2015)

A (regularized) finite Blaschke product B is decomposable as $B = g \circ f$ with Blaschke products f and g of degree $m \geq 2$ and $n \geq 2$, respectively, if and only if the critical points of B can be partitioned into multisets $A_0, A_1, \ldots, A_{n-1}$ such that:

(i) *The set A_0 contains $m - 1$ elements, and each set $A_1, \ldots, A_{n-1}$ contains m elements.*

(ii) *Two critical points of B have the same multiplicity whenever they belong to the same set A_k for some $k = 1, \ldots, n - 1$,*

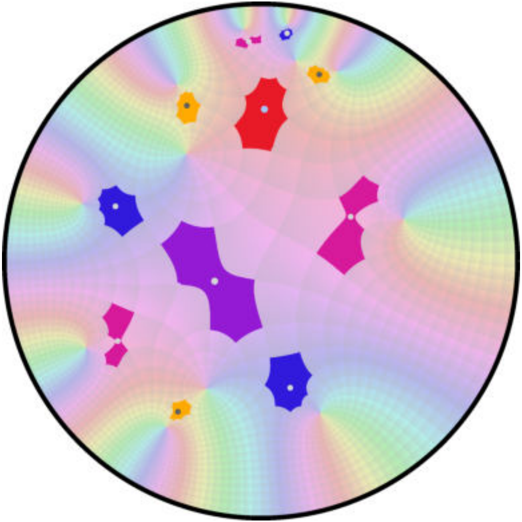
(iii) *Let f_0 be (one and then any) Blaschke product of degree m with A_0 as set of critical points. Then f_0 is constant on each A_k for $k = 1, \ldots, n - 1$.*

If these conditions are satisfied then B can be decomposed as $B = g_0 \circ f_0$, and the general form of such decompositions is

$$B = (g_0 \circ h^{-1}) \circ (h \circ f_0)$$

with a conformal disk automorphism h.

Checking the Conditions in the DGSSV-Theorem



(i) The partitioning of critical points can be read off from the color and the shape of the exceptional tiles.

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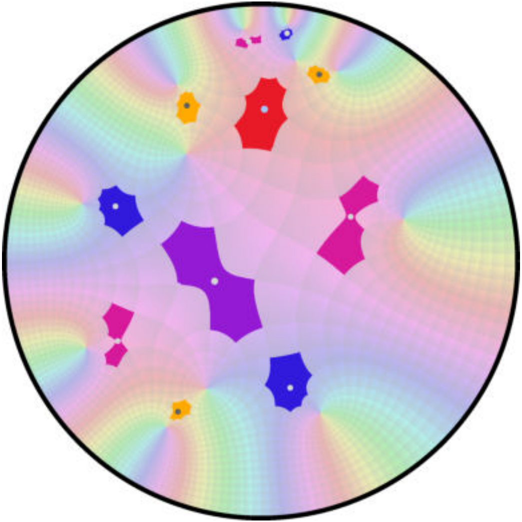

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The set A_0 has 2 elements

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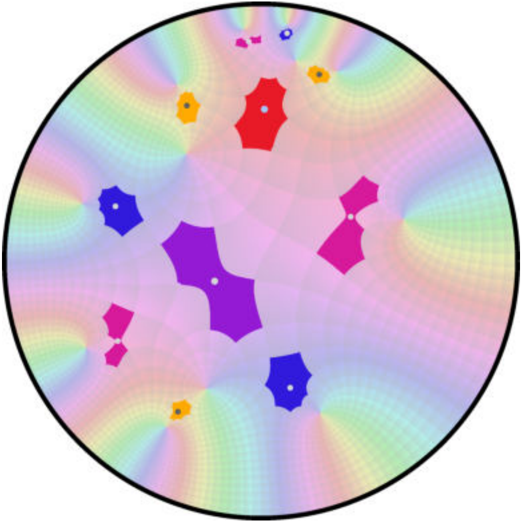
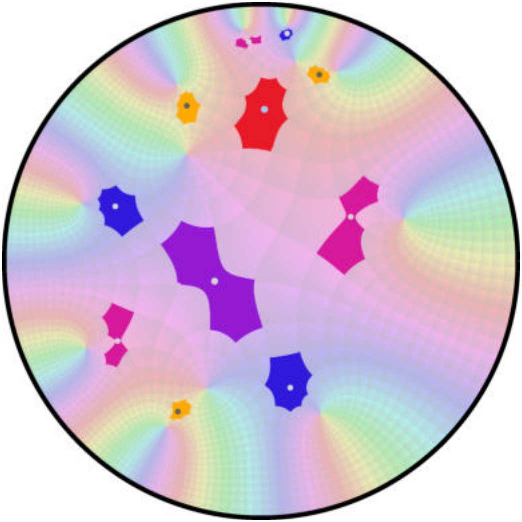


(i) The partitioning of critical points can be read off from the color and the shape of the exceptional tiles.

The set A_0 has 2 elements, each of the other sets A_1, A_2, A_3 has 3 elements.

Checking the Conditions in the DGSSV-Theorem



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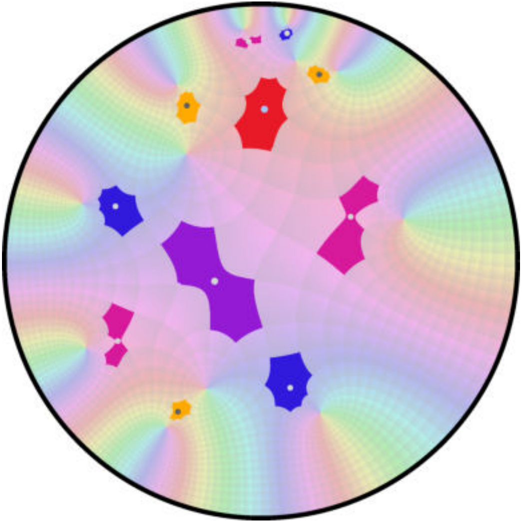


(i) The partitioning of critical points can be read off from the color and the shape of the exceptional tiles.

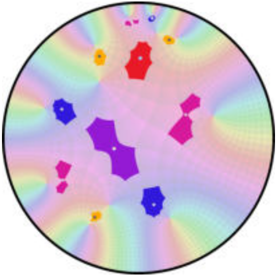
The set A_0 has 2 elements, each of the other sets A_1, A_2, A_3 has 3 elements.

(ii) All critical points have multiplicity 1.

(iii) How do we see that f_0 is constant on each set A_k?

Checking the conditions in the DGSSV-Theorem

Condition (iii) is equivalent to the fact that f_0 maps all exceptional tiles associated with the same set A_k onto one and the same tile. This is a matter of symmetry.



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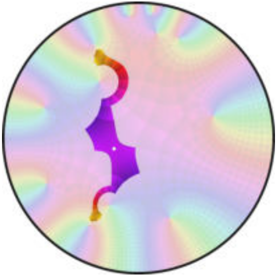

Knowing (or guessing) which exceptional tiles contain the critical points of f_0, this can be checked by constructing symmetric paths that connect the tiles in the corresponding set (as shown for the yellow tiles).

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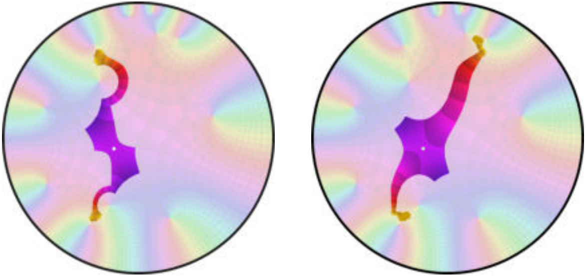

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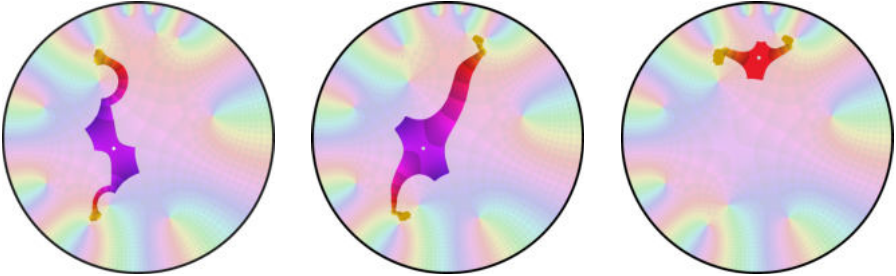

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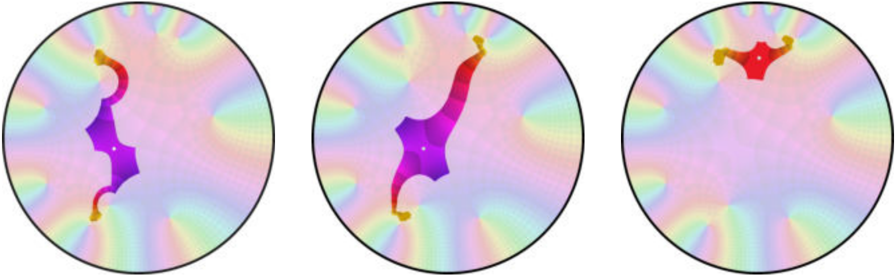

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All this holds up to some error depending on the resolution of the tiling.

A Theorem of Ritt

There is another, more abstract, criterion for decomposability of Blaschke products (originally stated for polynomials).

Theorem (Ritt, 1922)

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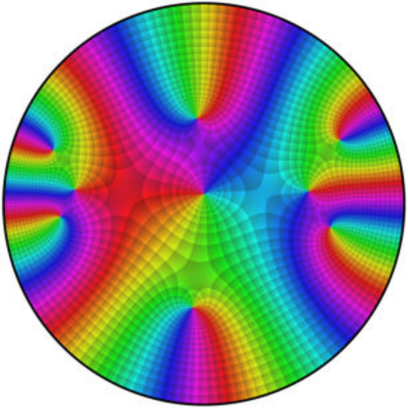
Can this partition be seen in the phase plot of a decomposable Blaschke product?

Visualizing Ritt's Theorem

This is a phase plot of $B = g \circ f$ with $m = \deg f = 3$ and $n = \deg g = 3$. Its monodromy group M_B acts on the sheets of the Riemann surface S_B.



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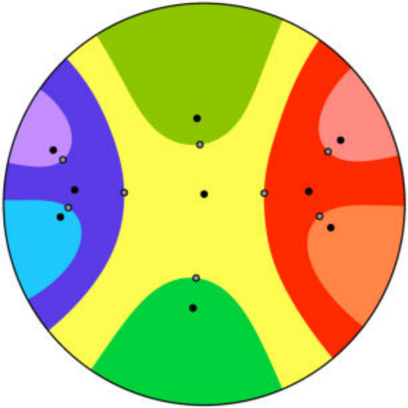

The sheets are associated with the (basins of) the zeros.

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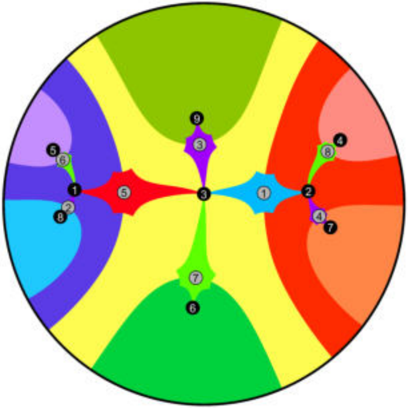

Here are the generators of M_B.

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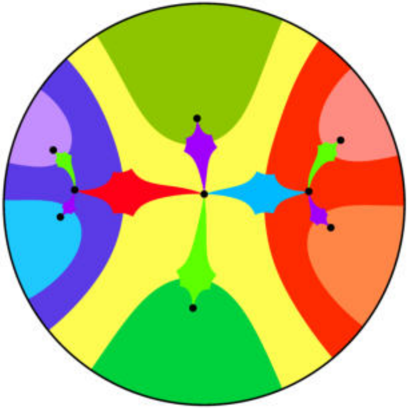

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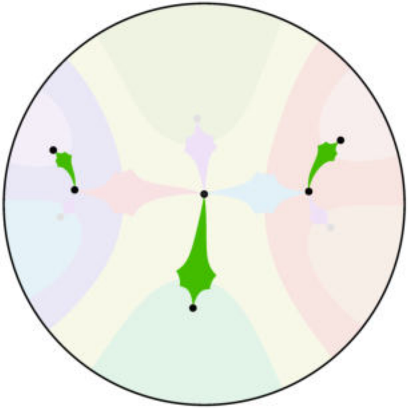

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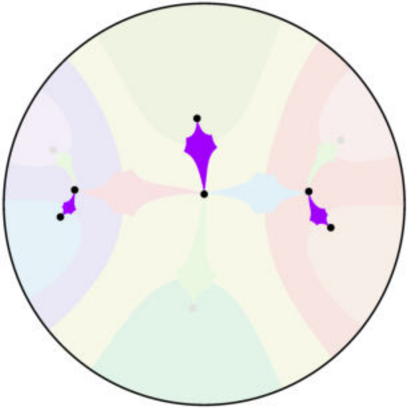

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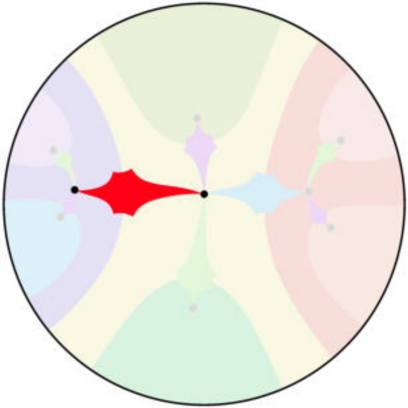

Here are the generators of M_B.
These three cells together
represent one generator,
these represent another one,
this single cell is the third one,

Visualizing Ritt's Theorem

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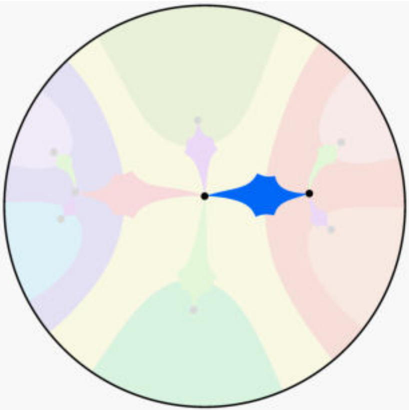

Here are the generators of M_B.
These three cells together
represent one generator,
these represent another one,
this single cell is the third one,
and this is the last one.

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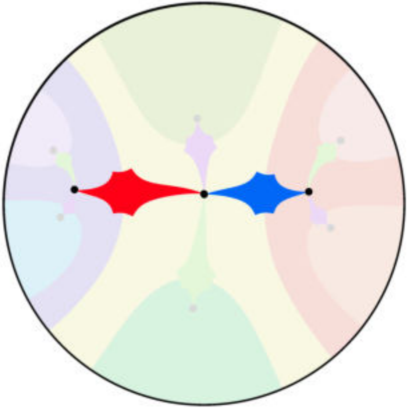

The last two are associated with critical points of f

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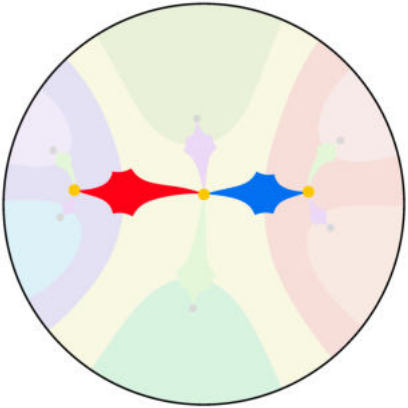

The last two are associated with critical points of f, these act on $m = 3$ zeros (sheets) of B

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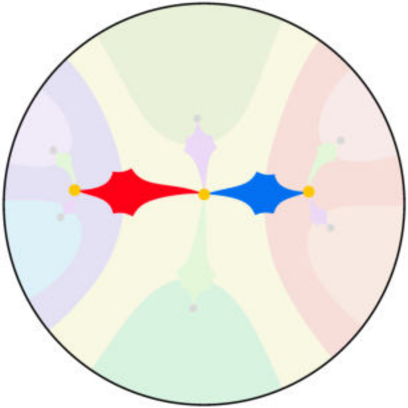

The last two are associated with critical points of f, these act on $m = 3$ zeros (sheets) of B (this is a non-trivial fact which follows from the transitivity of the monodromy group of f.)

Visualizing Ritt's Theorem

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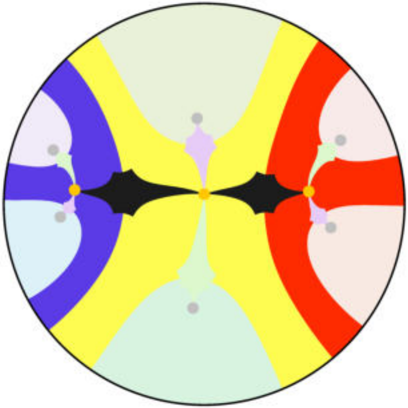

The last two are associated with critical points of f, these act on $m = 3$ zeros (sheets) of B (this is a non-trivial fact which follows from the transitivity of the monodromy group of f.) These zeros form the first group of the partition.

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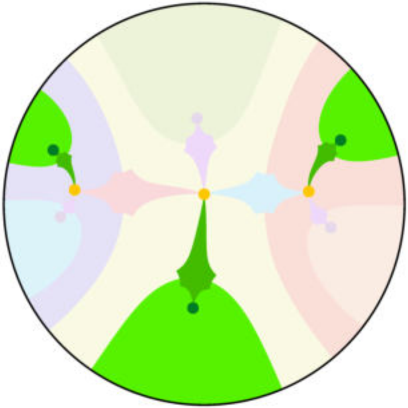

The last two are associated with critical points of f, these act on $m = 3$ zeros (sheets) of B (this is a non-trivial fact which follows from the transitivity of the monodromy group of f.) These zeros form the first group of the partition. Applying the remaining $n - 1$ generators successively to the m zeros in the first group, yields the members of the other groups.

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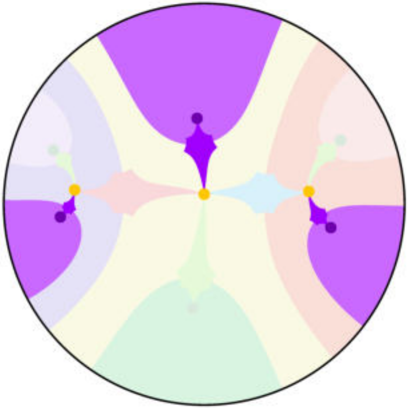

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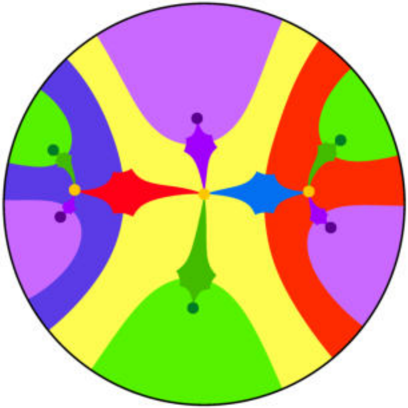

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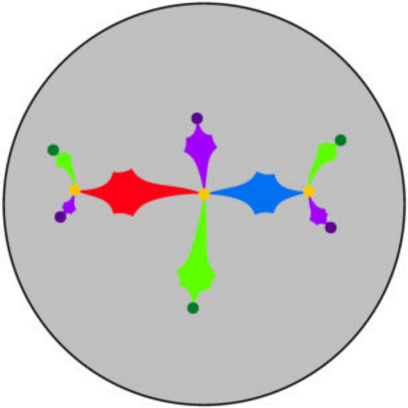

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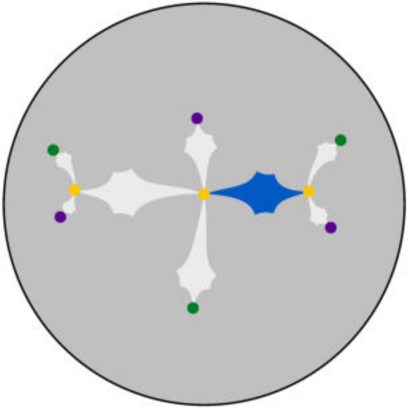

The generators of M_B associated with critical points of f respect the partition, since they act only inside the first (yellow) group.

Visualizing Ritt's Theorem

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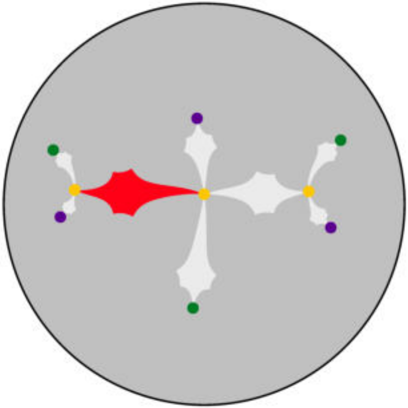

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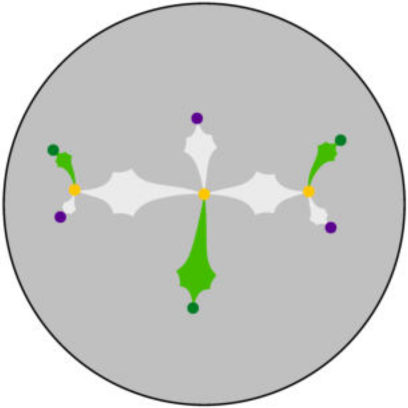

The generators of M_B associated with critical points of f respect the partition, since they act only inside the first (yellow) group. The generators of M_B associated with critical points of g respect the partition, since they permute the groups (yellow and green)

Visualizing Ritt's Theorem

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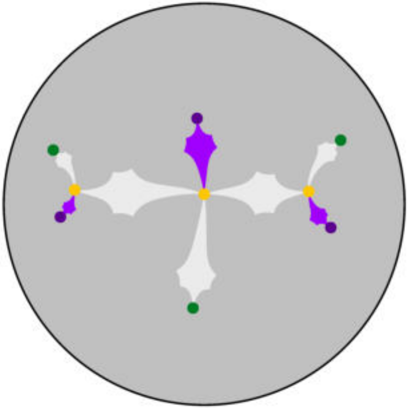

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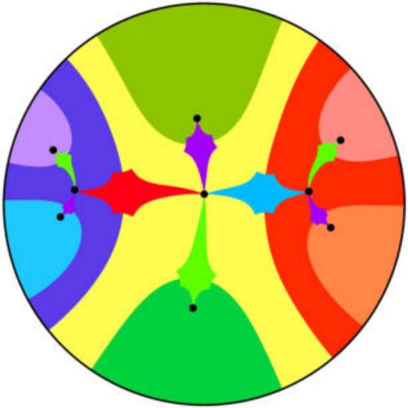

There is somewhat more to discover.

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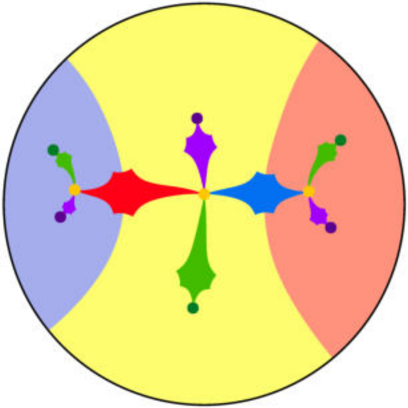

There is somewhat more to discover.
Each of the highlighted superbasins is mapped by f onto a copy of the unit disk. The generators of associated with f permute these.

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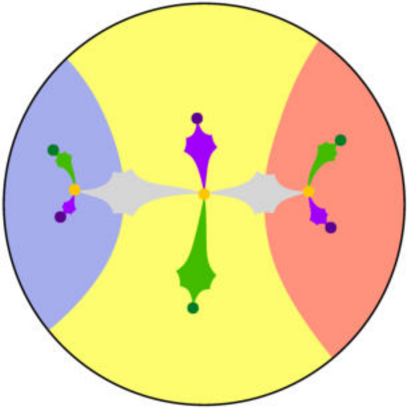

There is somewhat more to discover.
Each of the highlighted superbasins is mapped by f onto a copy of the unit disk. The generators of associated with f permute these.
The generators associated with g operate inside the basins, and they all act in the same way.

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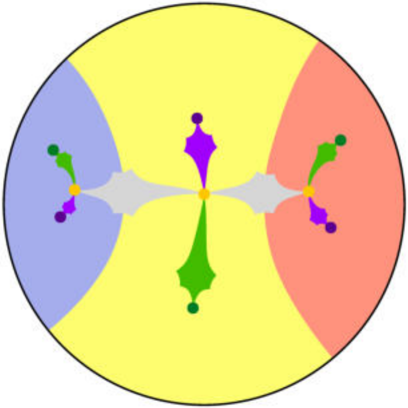

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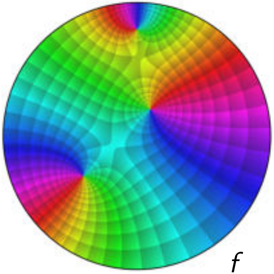
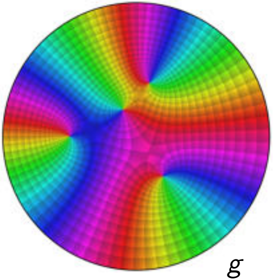
Let's look at another example.

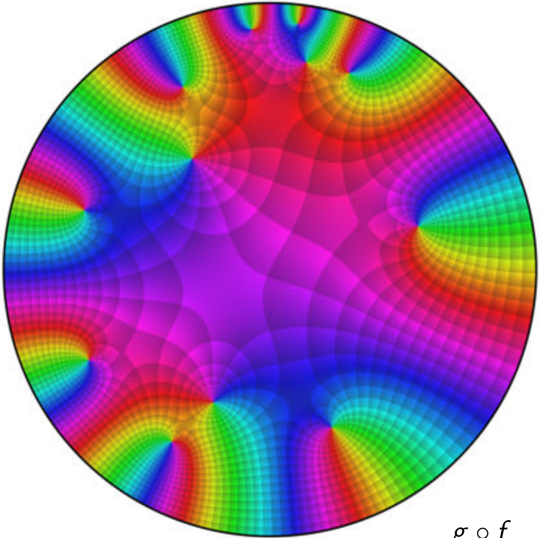
The monodromy group of a composition



f



g



$g \circ f$

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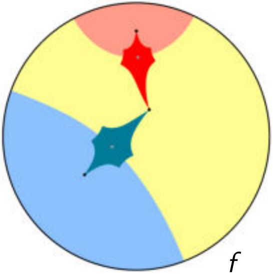
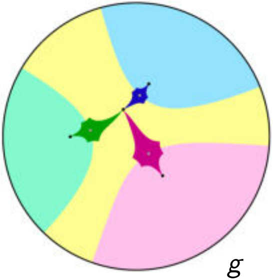
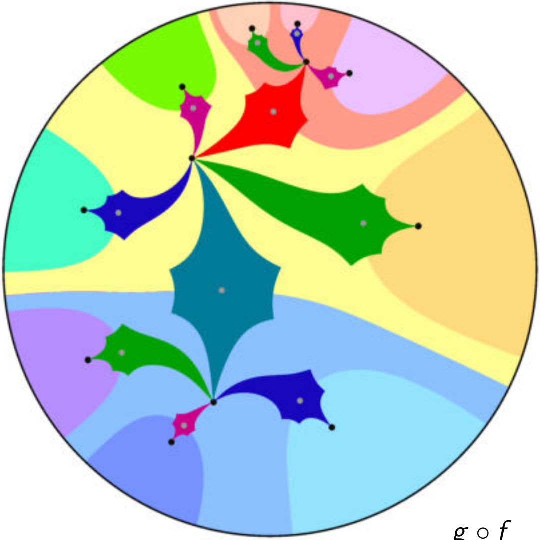


f



g



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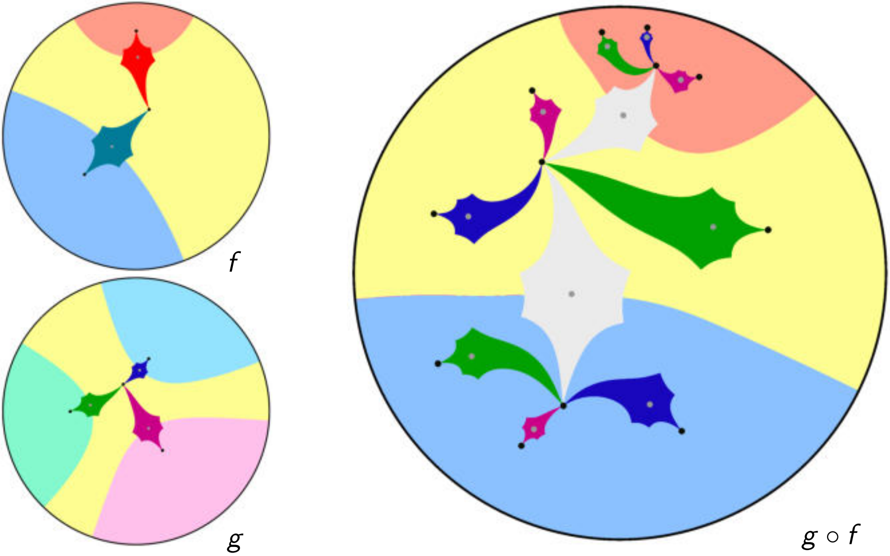

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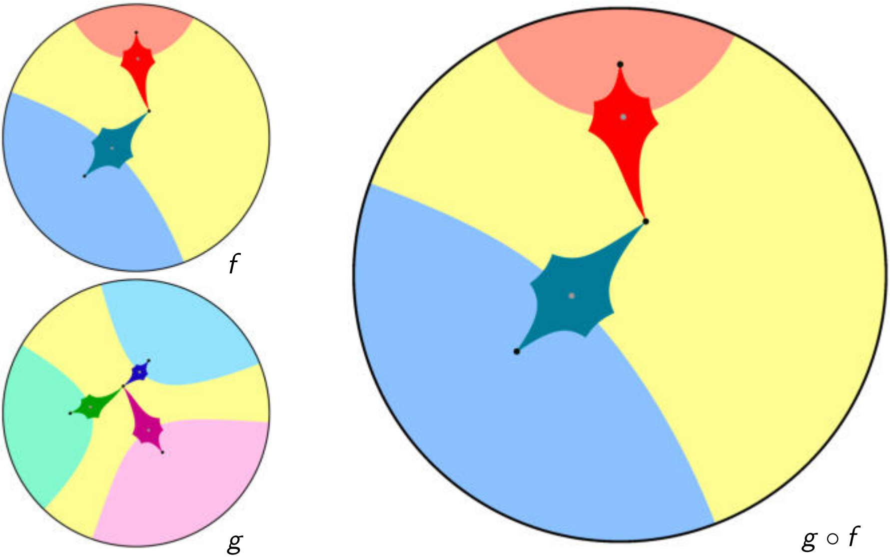

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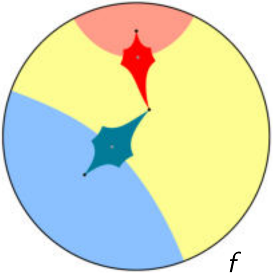
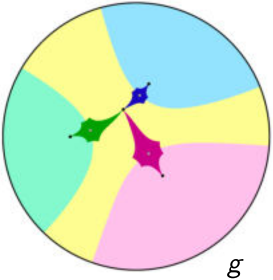


f



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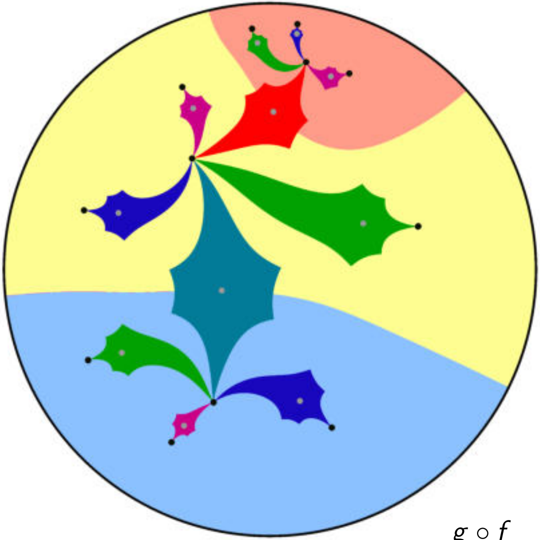


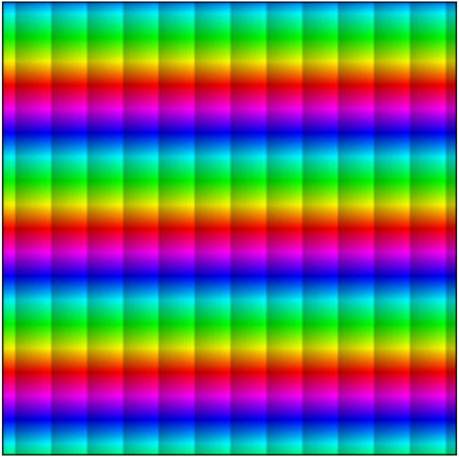
$g \circ f$

...is the direct product of the monodromy groups of its factors.

A Picture Book of Functions

Skip

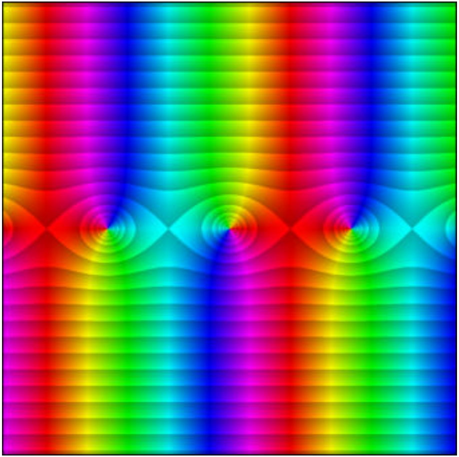
A picture book of complex functions



The exponential function

$$e^z = 1 + \frac{z}{1\,!} + \frac{z^2}{2\,!} + \ldots + \frac{z^k}{k\,!} + \ldots$$

A picture book of complex functions



The exponential function

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The sine function is a sum of two exponentials,

$$\sin z = \frac{1}{2\mathrm{i}} \left(e^{\mathrm{i}z} - e^{-\mathrm{i}z}\right)$$

A picture book of complex functions

The exponential function

$$e^z = 1 + \frac{z}{1\,!} + \frac{z^2}{2\,!} + \ldots + \frac{z^k}{k\,!} + \ldots$$

A linear combination of three exponential functions,

$$f(z) = \sum c_k e^{a_k z}$$

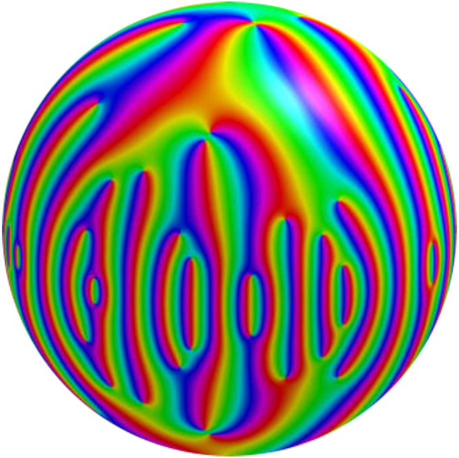
A picture book of complex functions

A finite Blaschke product

$$f(z) := \prod_{k=1}^{50} \frac{z - z_k}{1 - \overline{z_k} z}.$$

Wilhelm Blaschke (1885-1962)

A picture book of complex functions



A finite Blaschke product on the Riemann sphere

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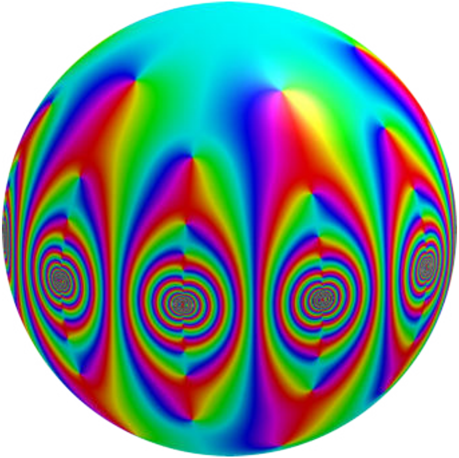

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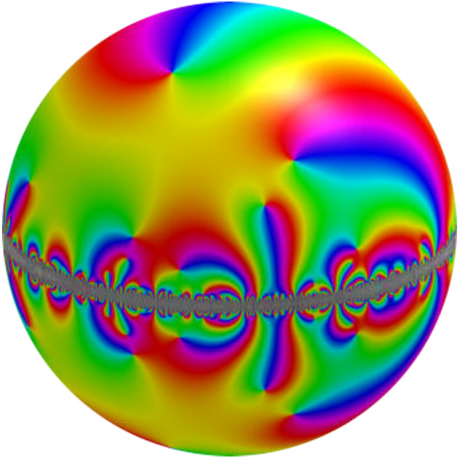

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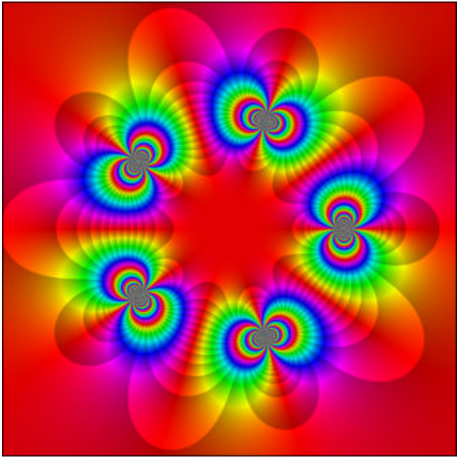
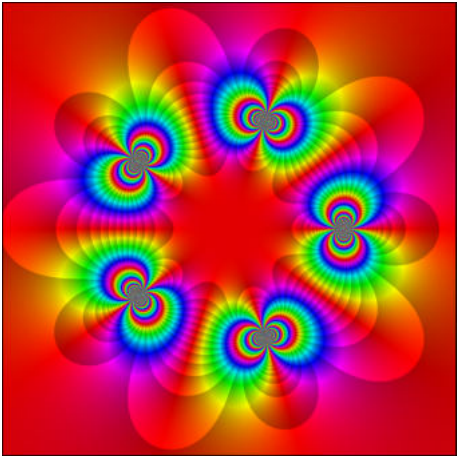


A singular inner function generated from an atomic measure at the fifth roots of unity,

$$f(z) = \prod_{k=1}^{5} \exp \frac{z + z_k}{z - z_k},$$

where $z_k = \omega^k$ with $\omega = \mathrm{e}^{2\pi\mathrm{i}/5}$.



A singular inner function generated from an atomic measure at the fifth roots of unity,

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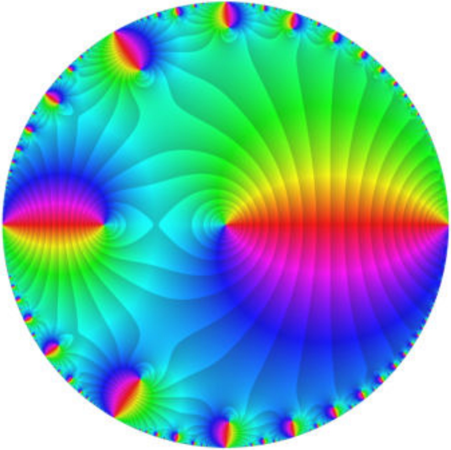
where $z_k = \omega^k$ with $\omega = \mathrm{e}^{2\pi\mathrm{i}/5}$.

This function has no zeros in the unit disk and constant modulus 1 almost everywhere on the unit circle.

A picture book of complex functions

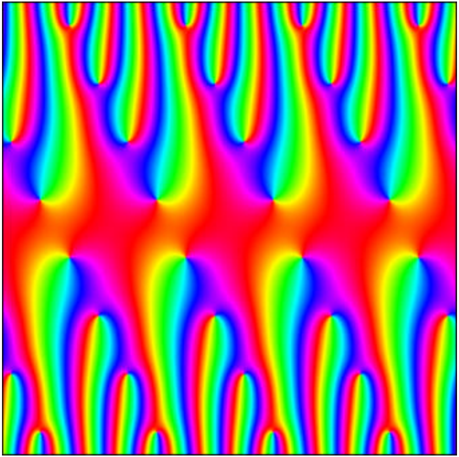


$$\frac{z}{1-z} + \frac{z^2}{1-z^2} + \ldots + \frac{z^n}{1-z^n} + \ldots$$

Johann Heinrich Lambert
(1728-1777)

The Lambert function is the generating function of the divisor function σ_0, its nth Taylor coefficient coincides with the number of divisors of n.

A picture book of complex functions

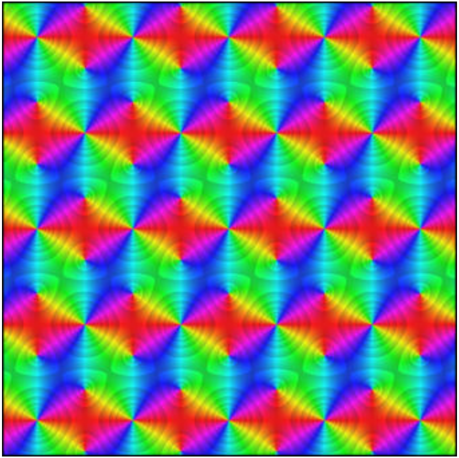


A Jacobi Theta function

$$f(z) := \sum_{k=-\infty}^{\infty} q^{k^2}\, \mathrm{e}^{2k\pi\mathrm{i}\, z}$$

Carl Gustav Jacobi
(1804-1851)

A picture book of complex functions

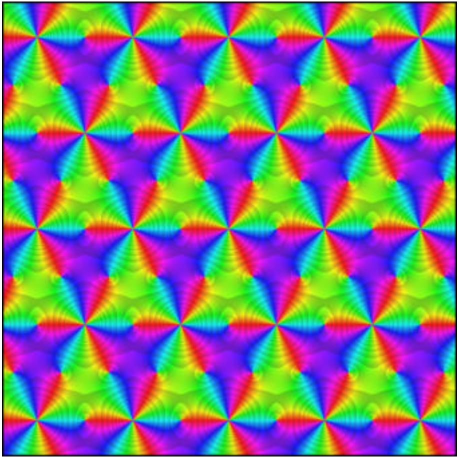


A Weierstrass' $\wp$–Function

$$f(z) = \frac{1}{z^2} + \sum_{p \in P, p \neq 0} \left[\frac{1}{(z-p)^2} - \frac{1}{p^2} \right]$$

Karl Weierstraß (1815-1897)

A picture book of complex functions



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$$f(z) = \frac{1}{z^2} + \sum_{p \in P, p \neq 0} \left[\frac{1}{(z-p)^2} - \frac{1}{p^2} \right]$$

and its derivative.

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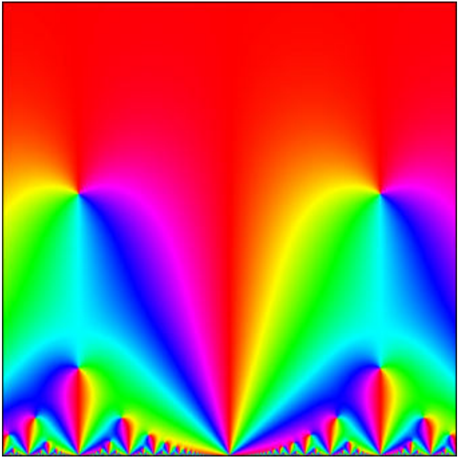


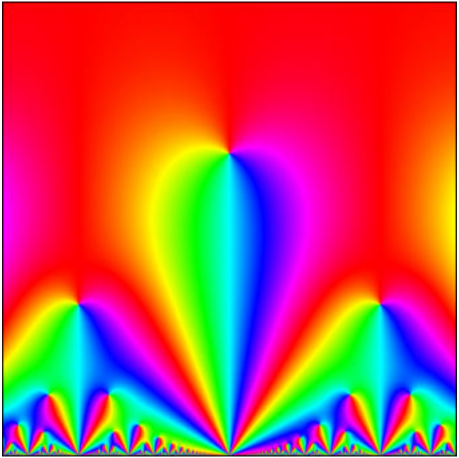
The Eisenstein series G_4

Eisenstein series

$$G_k(z) = \sum_{\substack{(c,d)\in\mathbb{Z}\times\mathbb{Z}\\(c,d)\neq(0,0)}} \frac{1}{(cz+d)^k}$$

Ferdinand Eisenstein (1823 – 1852)

A picture book of complex functions



The Eisenstein series G_6

Eisenstein series

$$G_k(z) = \sum_{\substack{(c,d)\in\mathbb{Z}\times\mathbb{Z} \\ (c,d)\neq(0,0)}} \frac{1}{(cz+d)^k}$$

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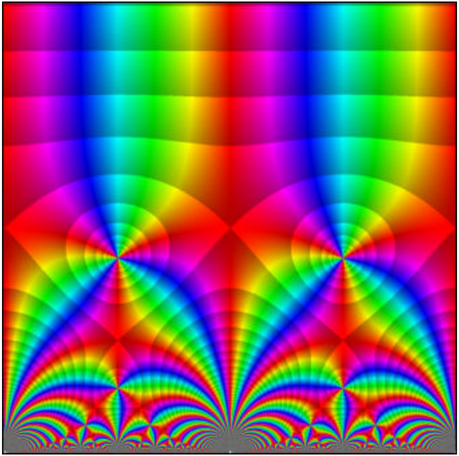
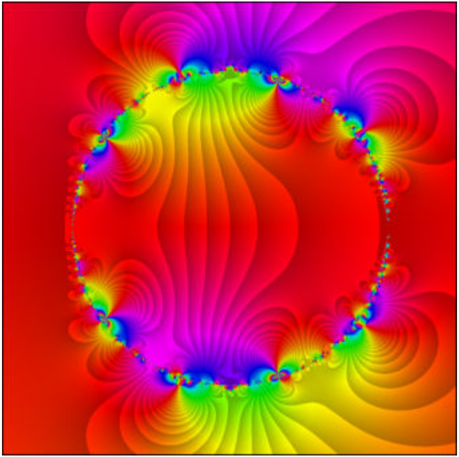


Klein's automorphic j-function

$$j(z) = 12^3 \, \frac{20\, G_4^3}{20\, G_4^3 - 49\, G_6^2}$$

Felix Klein (1849-1929)

A picture book of complex functions

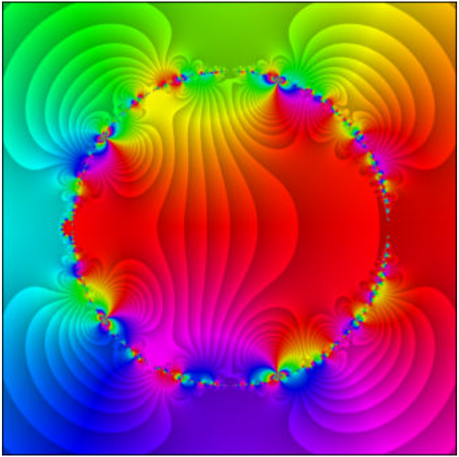


Ramanujan's continued fraction, convergent 200.

$$1+\cfrac{z}{1+\cfrac{z^2}{1+\cfrac{z^3}{1+\ddots}}}$$

Srinivasa Ramanujan (1887-1920)

A picture book of complex functions



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The Riemann Zeta function

$$f(z) = \frac{1}{1^z} + \frac{1}{2^z} + \frac{1}{3^z} + \frac{1}{4^z} \dots$$

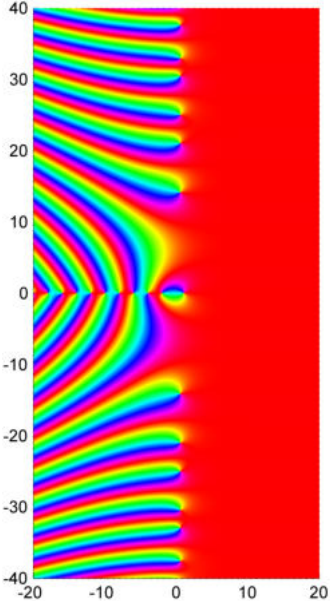
Bernhard Riemann (1826-1866)

Riemann's explicit formula

In his celebrated paper "Über die Anzahl der Primzahlen unter einer gegeben Größe" of 1859, Bernhard Riemann derives an explicit formula for analytic continuation of the Zeta function,

$$2\sin(\pi z)\,\Gamma(z)\,\zeta(z) = \mathrm{i} \oint_C \frac{(-x)^{z-1}}{e^x - 1}\,dx.$$

The curve C starts at $+\infty$, runs once around the origin in positive direction and returns to $+\infty$.



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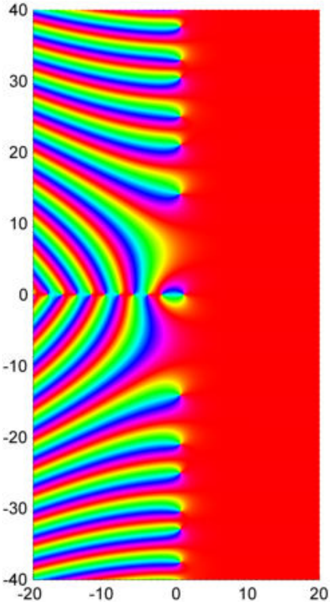
The curve C starts at $+\infty$, runs once around the origin in positive direction and returns to $+\infty$. This formula implies that Zeta has a simple pole at 1 and "trivial" zeros at $-2, -4, -6, \ldots$, but there are others, nicely aligned along $\operatorname{Re} z = 1/2$.



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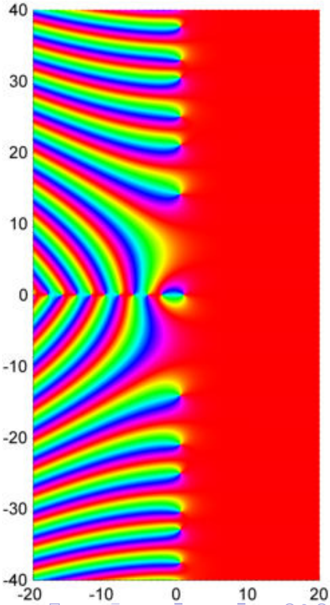
Are they ?



Non-trivial zeros of the Zeta function

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Moreover, he heuristically estimated the number $N(T)$ of non-trivial zeros which satisfy $0 < \operatorname{Im} z < T$ by

$$N(T) \approx \frac{T}{2\pi} \log \frac{T}{2\pi} - \frac{T}{2\pi}$$

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and claimed:

Man findet nun in der Tat etwa so viele reelle Wurzeln innerhalb dieser Grenzen, und es ist sehr wahrscheinlich, daß alle Wurzeln reell sind. Hiervon wäre allerdings ein strenger Beweis zu wünschen, ich habe indes die Aufsuchung desselben nach einigen flüchtigen vergeblichen Versuchen beiseite gelassen, da er für den nächsten Zweck meiner Untersuchungen entbehrlich schien.

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and claimed:
Indeed one finds about as many real roots within these bounds, and it is very likely that all roots are real. A strict proof of this fact would be desirable, however, after some unsuccessful attempts, I abandoned searching for one, because it was expendable for the next purpose of my investigations.

The Riemann Hypothesis

That the (nontrivial) zeros "are real" means in fact that they exactly lie in the middle of the critical strip, i.e., their real part equals $1/2$. This innocent statement is the celebrated Riemann Hypothesis.

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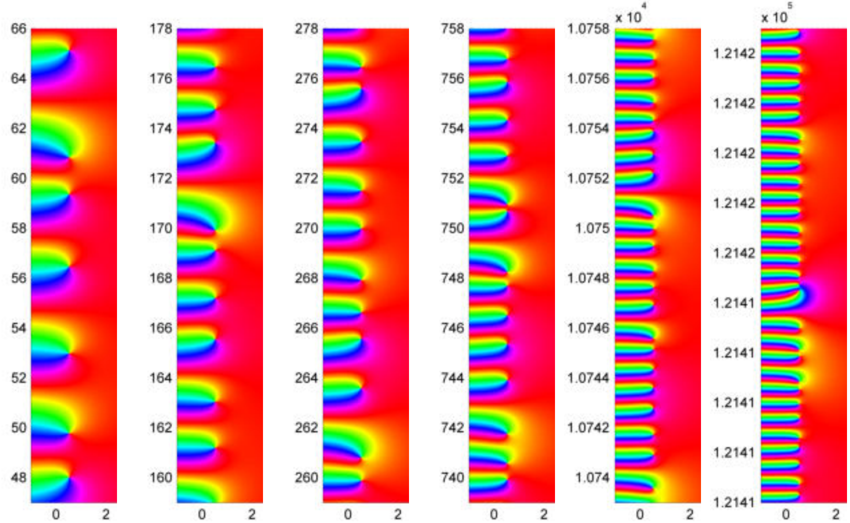
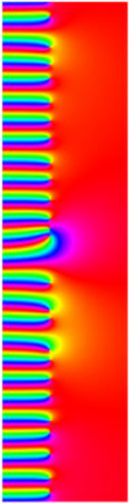
Though it is known today that more than 10 000 000 000 000 non-trivial zeros indeed lie on the critical line, the problem withstands all attacks and seems far from being solved.

Non-trivial zeros of the Zeta function



The Zeta function in the critical strip



The phase portrait of the Zeta function is surprisingly rich.

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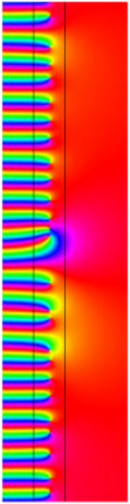

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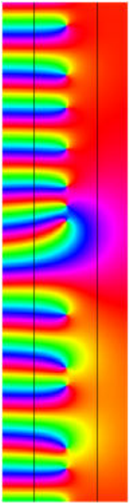

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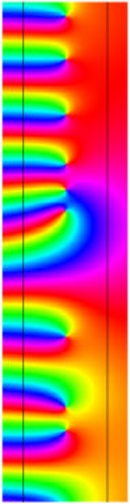

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Here we are in the critical strip at height $\operatorname{Im} z = 121\,415$.

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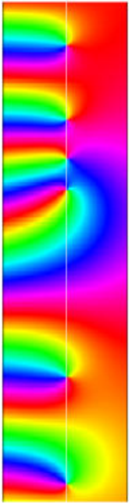

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Here we are in the critical strip at height $\operatorname{Im} z = 121\,415$.
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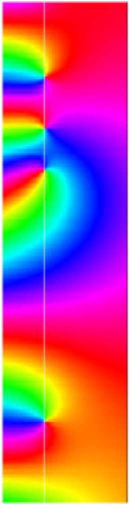
In particular the right half of the critical strip is remarkably colorful.

The Zeta function in the critical strip



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In order to explore this region further we send out scouts.

Strings and their chromatic numbers $\mathrm{chrom}\, S$

A string is a simple closed curve (Jordan curve) which fits into an open strip of width $1/2$.

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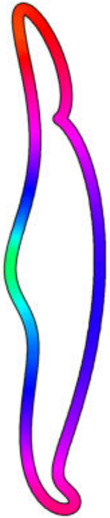

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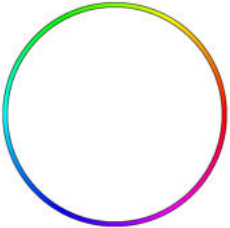
Strings are colored continuously by colors of the `hsv`–color wheel.



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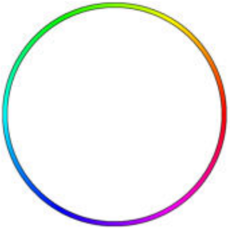

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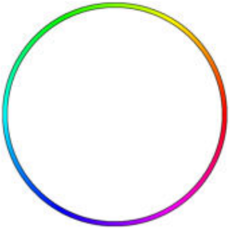
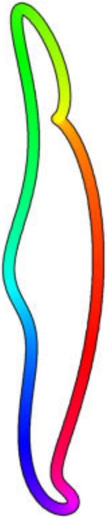


$\mathrm{chrom}\, S = 0$

Strings and their chromatic numbers $\operatorname{chrom} S$



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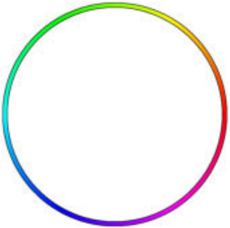

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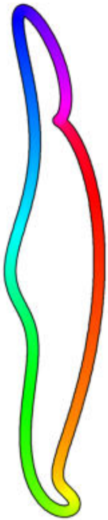

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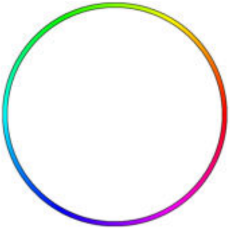

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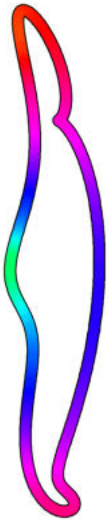

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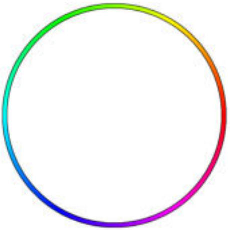
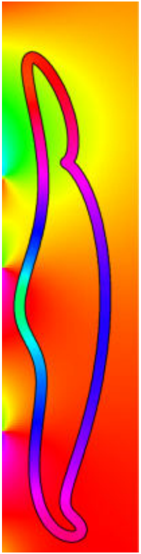


$\mathrm{chrom}\, S = 0$

Universality of the Zeta function



Strings live in the right half of the critical strip.

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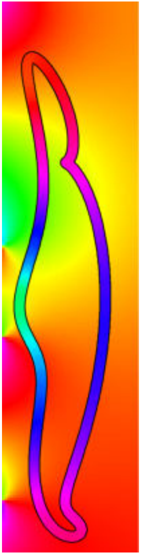

Strings live in the right half of the critical strip.
They are allowed to move in vertical direction.
Strings try to find a place where they become invisible.

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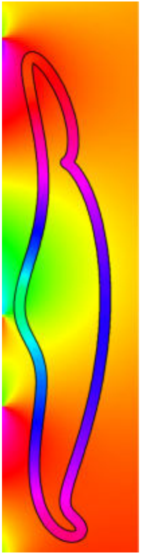

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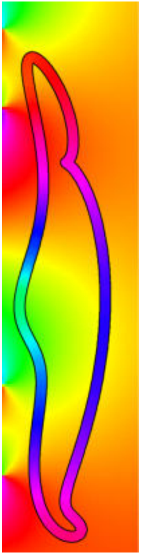

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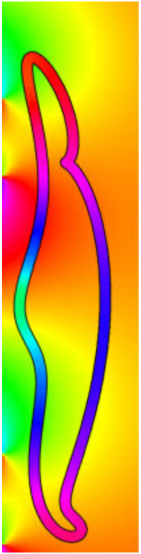

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We say that a string can hide itself if it can find a position where its colors differ from the background by an arbitrarily small amount.

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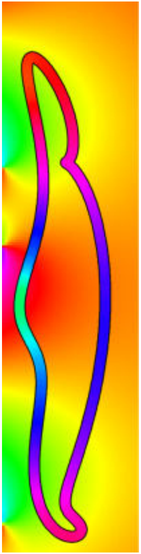

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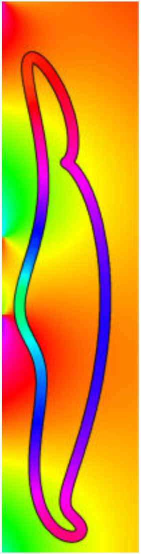

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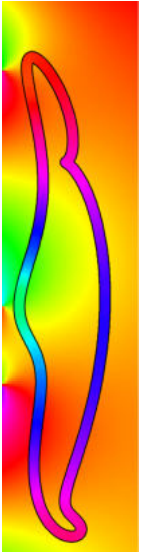

Strings live in the right half of the critical strip.
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Which strings can hide themselves in the phase portrait of the Riemann Zeta function ?

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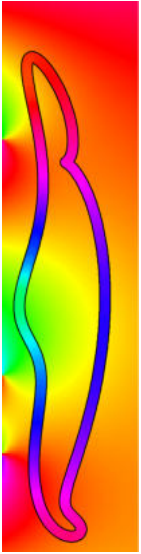

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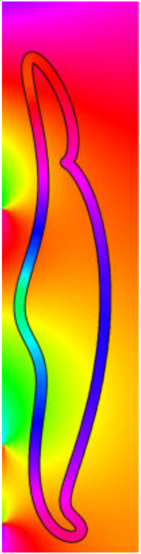

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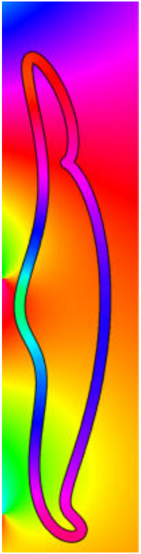

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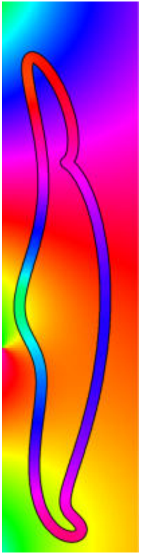

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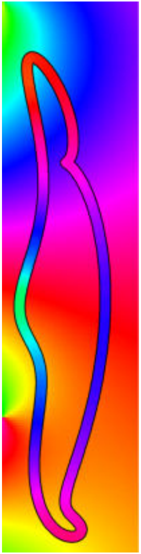

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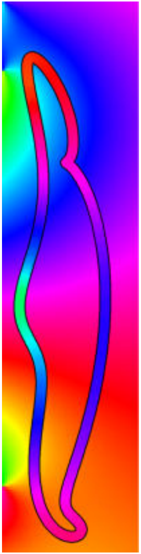

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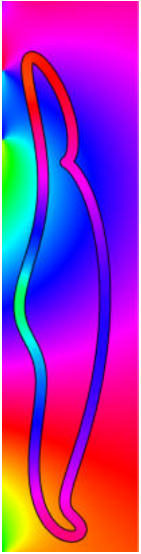

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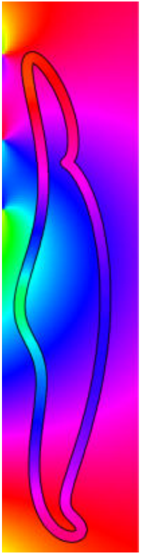

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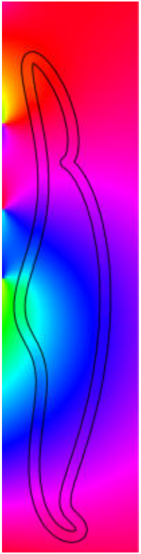

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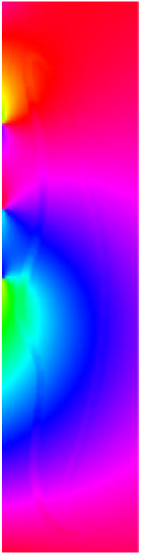

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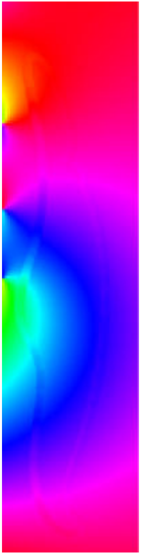

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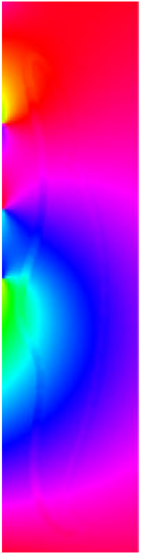

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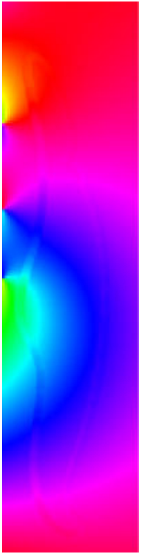
Theorem

Every string with chromatic number zero can hide itself in the phase plot of the Riemann Zeta function.

Universality of the Zeta function



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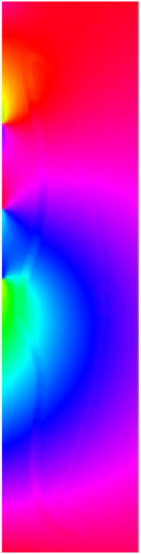

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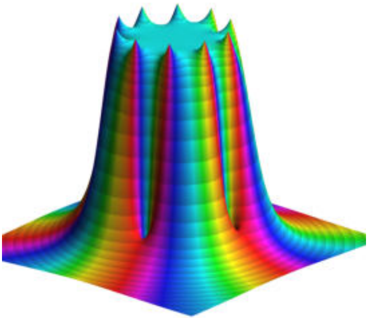
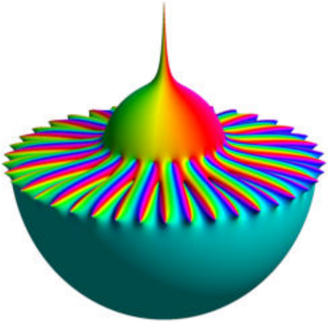
The necessity of the condition $\operatorname{chrom} S = 0$ is equivalent to the Riemann Hypothesis.

Software: the complex function explorer





www.mathworks.com/matlabcentral/fileexchange/45464-complex-function-explorer

www.mathworks.com/matlabcentral/fileexchange/44375-phase-plots-of-complex-functions

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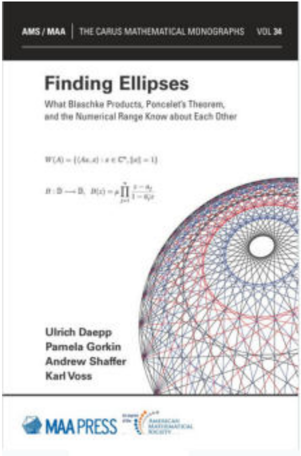
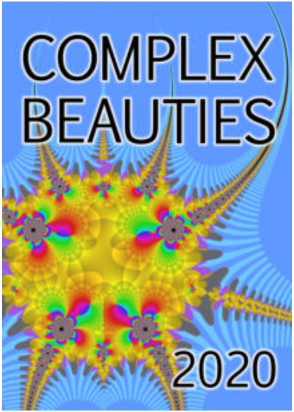
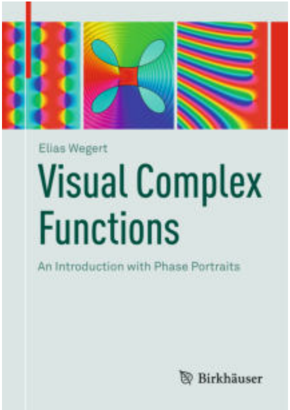


Learn more about phase plots





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