Seeing the Monodromy Group of a Blaschke Product
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Visualizing Complex Functions
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\[ G_f := \{(z, f(z)) \in \mathbb{C}^2 : z \in D\} \]

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This picture of the complex Gamma function, published 1909 in the famous book by Jahnke and Emde, acquired an almost iconic status.
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The analytic landscape depicts only the absolute value of a function and neglects its argument (phase). Jahnke and Emde compensated this by drawing lines of constant argument.
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Today analytic landscapes can be computed easily, and coloring allows one also to incorporate the argument.
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Instead of the argument one better uses the (well-defined) phase

\[ f(z)/|f(z)|. \]

It lives on the unit circle \( \mathbb{T} \), and can be encoded using a circular color scheme.
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Phase plots are special variants of domain coloring.
Phase plots outperform analytic landscapes

The phase plot of a function shows many properties more clearly than the analytic landscape.

An analytic landscape of $f(z) = e^{1/z}$
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An analytic landscape of $f(z) = e^{1/z}$ and its phase plot.
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An analytic landscape of $f(z) = e^{1/z}$ and its phase plot.

A function which is meromorphic in an open connected set (domain) $G$ is uniquely determined up to a positive constant factor by its phase plot.
Phase Plots:  Less is more
Phase Plots: Less is more – more or less
We illustrate the construction of a phase plot with the rational function $f(z) = (z - 1)/(z^2 + z + 1)$ in the square $|\Re z| \leq 2$, $|\Im z| \leq 2$. 
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All points of the \(w\)-plane with the same argument get the same color.
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All points of the $w$-plane with the same argument get the same color. Then every point $z$ in the domain of definition is colored like its image point $w = f(z)$. 
Phase plots and their modifications

We illustrate the construction of a phase plot with the rational function
\[ f(z) = \frac{z - 1}{z^2 + z + 1} \]
in the square \(|\text{Re } z| \leq 2, \ |\text{Im } z| \leq 2\).

Modifications of the color scheme in the \( w \)-plane allow one to read off properties of the function more easily. This version incorporates the absolute value of \( f \) by highlighting some contour lines of \(|f|\).
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Modifications of the color scheme in the \( w \)-plane allow one to read off properties of the function more easily. This variant demonstrates that the mapping \( f \) is conformal. With a few exceptions, all “tiles” have four right angled corners.
Phase plots and their modifications

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Classical domain coloring (Frank Farris) uses a two-dimensional color scheme, with brightness corresponding to absolute value, to encode the values of $f$ completely.
How to read it
Zeros and poles

Both rows show phase portraits of the power functions $f(z) = z^k$ for $k = 0$ (left), $k = 1, 2, 3$ (above) and $k = -1, -2, -3$ (below).
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Zeros and poles can be distinguished by the orientation of colors, their multiplicity can be read off easily.
Isochromatic lines and contour lines

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The isochromatic lines (sets with equal phase of \( f \)) and the contour lines (sets with equal absolute value of \( f \)) are perpendicular.

For the exponential function both families consist of parallel lines. Here we see \( f(z) = \exp(5z) \) in \(|\Re z| < 5, |\Im z| < 5\).
Critical points $\zeta$ of a function $f$ are the zeros of its derivative. Points where $f'(\zeta) = 0$ and $f(\zeta) \neq 0$ are called saddle points.
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In the phase plot of $f$ saddle points are the only crossing points of isochromatic lines.
The **order** of a saddle point is the multiplicity of the zero of $f'$. A saddle point of order $n$ is the crossing of $n + 1$ isochromatic lines.

The saddle points in these phase plots have orders 1, 2, 3 and 8.
The Order of Saddle Points

A tile containing saddle points is called exceptional. When it has saddle points of orders summing up to $n$, it has $4(n + 1)$ corners.

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Blaschke Products
A Blaschke factor is a Moebius transformation of the form

\[ f(z) = c \frac{z - z_0}{1 - \overline{z_0}z}, \quad |z_0| < 1, \quad |c| = 1. \]
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A Blaschke product is the product of Blaschke factors,

\[ B(z) = c \prod_{k=1}^{n} \frac{z - z_k}{1 - \overline{z}_k z}, \]

it has modulus 1 on \( \mathbb{T} \).

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Due to \( B(1/\bar{z}) = 1/\overline{B(z)} \), the phase plot of Blaschke products on the Riemann sphere is symmetric with respect to the equator.
Intermezzo: The Phase Flow
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The phase flow

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This can be modeled by a vector field. If \( f : D \to \hat{\mathbb{C}} := \mathbb{C} \cup \{\infty\} \) is a meromorphic function, then \( V_f \) defined by

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V_f(z) := -\frac{f(z) \overline{f'(z)}}{|f(z)|^2 + |f'(z)|^2}
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is smooth on \( D \), and \( V_f(z) \) is tangent to the isochromatic lines of \( f \) at \( z \) (with \( \mathbb{C} \cong \mathbb{R}^2 \)).
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The vector field \( V_f \) generates a continuous semigroup, the phase flow \( \Psi_f \).
Visualization of the proper phase flow is demanding, here is a cheap substitute. It has the same orbits but a different (discontinuous) speed.

The animated phase plot is a pull-back of the range disk, covered by a rotating polar chessboard mask.
Basins of zeros

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The phase flow of a Blaschke product $B$ is special: removing from $\mathbb{D}$ all **stable manifolds** of the saddle points, the remaining set $\mathbb{D} \setminus S$ is the disjoint union of simply connected domains, which are the **basins of zeros**.
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This statement must be modified somewhat when $B$ has multiple zeros.
The basins of attraction of the zeros of $B$ are natural candidates to form the sheets of the Riemann surface of $B^{-1}$. For the following constructions we assume that $B$ is regularized, i.e.,

1. all zeros of $B$ are simple (which implies that $0$ is not a critical value),
2. if $\zeta_j$ and $\zeta_k$ are critical points of $B$, then
   
   \[ |B(\zeta_j)| = |B(\zeta_k)| \implies B(\zeta_j) = B(\zeta_k) \]
   
   \[ \frac{B(\zeta_j)}{B(\zeta_k)} \in \mathbb{R}_+ \implies B(\zeta_j) = B(\zeta_k). \]

These are formal restrictions – they can always be achieved by replacing $B$ by $\tilde{B} = B_2 \circ B \circ B_1$, where $B_1$ and $B_2$ are appropriate Blaschke products of degree 1 (conformal automorphisms of $\mathbb{D}$). This transformation has no influence on the structures we are interested in.
The Riemann Surface of $B^{-1}$
A Blaschke product $B : \mathbb{D} \to \mathbb{D}$ of degree $n$ is an $n$-fold covering map.
Blaschke products as covering maps

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Blaschke products as covering maps

The phase flow allows us to determine the basins of the zeros.
Blaschke products as covering maps

Each basin is mapped onto a slit disk.
Since the Blaschke product is an $n$-fold covering map of $\mathbb{D}$ onto itself, its inverse $B^{-1}$ lives on a Riemann surface $S_B$ formed by $n$ sheets $D_1, \ldots, D_n$, where each sheet is a copy of $\mathbb{D}$. 
Riemann surfaces of Blaschke products

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Without loss of generality we may assume that \( B \) is regularized. Then the basins of attraction \( B_k \) of the zeros \( z_k \) of \( B \) are mapped bijectively onto disks \( D_k \) with radial slits. These are the sheets \( D_k \) of the Riemann surface \( S_B \).
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The neighboring relations of the basins tell us how the sheets have to be glued along their slits.
Blaschke Products: Monodromy
Let $W = B(S)$ be the set of critical values of a regularized Blaschke product $B$, and consider closed oriented paths (loops) $\gamma$ in $\hat{D} := \mathbb{D} \setminus W$ with base point $0 \notin W$. 
Let $W = B(S)$ be the set of critical values of a regularized Blaschke product $B$, and consider closed oriented paths (loops) $\gamma$ in $\hat{\mathbb{D}} := \mathbb{D} \setminus W$ with base point $0 \notin W$.

These loops form a group with respect to concatenation.

The fundamental group $\pi_1(\hat{\mathbb{D}})$ of $\hat{\mathbb{D}}$ consists of equivalence classes $[\gamma]$ of homotopic loops in $\hat{\mathbb{D}}$. 
Any path \( \gamma \) in \( \mathring{D} \) can be \emph{lifted} to a path \( \Gamma \) on the Riemann surface \( S_B \).
Monodromy Group of a Blaschke Product

Any path $\gamma$ in $\hat{D}$ can be \textit{lifted} to a path $\Gamma$ on the Riemann surface $S_B$.

The initial point of $\Gamma$ can be chosen on any sheet ("above" the initial point of $\gamma$); once this point is fixed, $\Gamma$ is uniquely determined.
Monodromy Group of a Blaschke Product

Any path $\gamma$ in $\hat{D}$ can be *lifted* to a path $\Gamma$ on the Riemann surface $S_B$. The initial point of $\Gamma$ can be chosen on any sheet ("above" the initial point of $\gamma$); once this point is fixed, $\Gamma$ is uniquely determined. If $\gamma$ is a loop, this need not be so for $\Gamma$, since the sheet of its terminal point can be different from the sheet of its initial point.
Any path $\gamma$ in $\hat{\mathbb{D}}$ can be \textit{lifted} to a path $\Gamma$ on the Riemann surface $S_B$. Denoting by $D_j$ the sheet containing the initial point of $\Gamma$, and by $D_k$ the sheet containing its terminal point, this defines a permutation

$$M_\gamma : j \mapsto k, \quad j = 1, \ldots, n.$$
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Denoting by \( D_j \) the sheet containing the initial point of \( \Gamma \), and by \( D_k \) the sheet containing its terminal point, this defines a \textit{permutation}

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M_\gamma : j \mapsto k, \quad j = 1, \ldots, n.
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Since \( M_\gamma \) depends only on the homotopy class of \( \gamma \), we write \( M_{[\gamma]} \).
Monodromy Group of a Blaschke Product

Any path $\gamma$ in $\hat{D}$ can be *lifted* to a path $\Gamma$ on the Riemann surface $S_B$. This defines the *monodromy mapping*

$$M_B : [\gamma] \mapsto M_{[\gamma]}.$$  

$M_B([\gamma])$ is the *permutation of sheets* of $S_B$ induced by the lifting of a closed loop $\gamma$. In the image on the left

$$M_B([\gamma]) = (1\ 2\ 3).$$
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Endowed with concatenation of loops, $M_B : \pi_1(\hat{D}) \to S_n$ is the monodromy group of $B$, a subgroup of the symmetric group $S_n$. 
Generators of the monodromy group

The fundamental group $\pi_1(\hat{\mathbb{D}})$ is generated by the (equivalence classes of) "small loops" $\gamma_j$ around the critical values $B(\zeta_j)$, and the monodromy group $M_B$ is generated by the permutations of the sheets $D_1, \ldots, D_n$ induced by these loops. This can be seen in the phase plot of $B$!
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These cells are the generators of the monodromy group $M_B$. 
Some examples

Phase plot of a Blaschke product (left), generators of its monodromy group (middle), and loops from which the cells are generated (right).

This Blaschke product has degree 3, with two saddle points of order 1. The generators of its monodromy group are (1 2) and (2 3), and $M_B$ is the symmetric group $S_3$. 
Some examples

Phase plot of a Blaschke product (left), generators of its monodromy group (middle), and loops from which the cells are generated (right).

This Blaschke product has degree 4, with a saddle point of order 3. The generator of its monodromy group is \((1\ 2\ 3\ 4)\), so that \(M_B = \mathbb{Z}_4\). The (only) critical value is very small.
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Phase plot of a Blaschke product (left), generators of its monodromy group (middle), and loops from which the cells are generated (right).

This Blaschke product has degree 4, with a saddle point of order 3. The generator of its monodromy group is $(1\ 2\ 3\ 4)$, so that $M_B = \mathbb{Z}_4$. The (only) critical value is very small, a zoom-in shows the loop more clearly.
Some examples

Phase plot of a Blaschke product (left), generators of its monodromy group (middle), and loops from which the cells are generated (right).

A **generic** Blaschke product of degree 5 has four saddle points of order 1. The generators of its monodromy group are (1 5), (2 5), (3 5), (4 5), and the monodromy group is the **symmetric group** $\mathfrak{S}_5$. 
Some examples

Phase plot of a Blaschke product (left), generators of its monodromy group (middle), and loops from which the cells are generated (right).

This Blaschke product has degree 4 and 3 saddle points $\zeta_1, \zeta_2, \zeta_3$ of order 1, but two critical values coincide, $B(\zeta_2) = B(\zeta_3) =: w$. 
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Phase plot of a Blaschke product (left), generators of its monodromy group (middle), and loops from which the cells are generated (right).

This Blaschke product has degree 4 and 3 saddle points $\zeta_1, \zeta_2, \zeta_3$ of order 1, but two critical values coincide, $B(\zeta_2) = B(\zeta_3) =: w$. Since a loop which encircles $w$ affects both cells $C_2$ and $C_3$, they “act simultaneously”, which results in the permutation $(13)(24)$. Together with the second generator $(34)$ this produces the monodromy group of $B$, which is the dihedral group $\mathbb{D}_4$. 
Some examples

Phase plot of a Blaschke product (left), generators of its monodromy group (middle), and loops from which the cells are generated (right).

Now we have $\deg B = 12$ with 11 saddle points, but only 5 different critical values: $w_1 = w_2 = w_3$, $w_5 = w_6 = w_7$ and $w_9 = w_{10} = w_{11}$. 
Some examples

Phase plot of a Blaschke product (left), generators of its monodromy group (middle), and loops from which the cells are generated (right).

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$(1, 10)(2, 11)(3, 12)$, $(10, 12)$, $(7, 10)(8, 11)(9, 12)$, $(11, 12)$, $(4, 10)(5, 12)(6, 11)$
Some examples

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Blaschke Products: Composition
The Blaschke product $B$ in the last example was special, because it was a composition of two Blaschke products of lower degree, $B = g \circ f$.

The figure illustrates how Blaschke products $f$ of degree 3 and $g$ of degree 4 are composed to a Blaschke product of degree 12.
Compositions of Blaschke Products

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The figure illustrates how Blaschke products $f$ of degree 3 and $g$ of degree 4 are composed to a Blaschke product of degree 12. Since a phase plot is constructed by pulling back the structure from the range plane to the domain, it should be read from right to left.
Compositions of Blaschke Products

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The phase plot of $g$ (middle) shows the critical points of $g$. 
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The phase plot of $g$ (middle) shows the critical points of $g$. By the chain rule,

$$ (g \circ f)' = (g' \circ f) \cdot f', $$

their pull back via $f$ are critical points of $B$ (left).
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The phase plot of $g$ (middle) shows the critical points of $g$. By the chain rule,

$$(g \circ f)' = (g' \circ f) \cdot f',$$

their pull back via $f$ are critical points of $B$ (left). The remaining critical points of $B$ are the critical points of $f$. 
The Blaschke product $B$ in the last example was special, because it was a composition of two Blaschke products of lower degree, $B = g \circ f$.

This can also be seen in the corresponding exceptional tiles.
Compositions of Blaschke Products

The Blaschke product $B$ in the last example was special, because it was a composition of two Blaschke products of lower degree, $B = g \circ f$.

This can also be seen in the corresponding exceptional tiles. Since $f$ maps $\mathbb{D}$ onto a 3-fold covering of $\mathbb{D}$, each exceptional tile of $g$ is triplicated in the phase plot of $B$. 

![Phase plots for $f$, $g$, and $B$.]
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This can also be seen in the corresponding exceptional tiles. Since $f$ maps $\mathbb{D}$ onto a 3-fold covering of $\mathbb{D}$, each exceptional tile of $g$ is triplicated in the phase plot of $B$. All tiles with the same color are conformally equivalent, since they are pulled back from the same tile in the image on the right.
A Criterion for Decomposability

Theorem (Daepp, Gorkin, Shaffer, Sokolowsky, Voss, 2015)

A (regularized) finite Blaschke product $B$ is decomposable as $B = g \circ f$ with Blaschke products $f$ and $g$ of degree $m \geq 2$ and $n \geq 2$, respectively, if and only if the critical points of $B$ can be partitioned into multisets $A_0, A_1, \ldots, A_{n-1}$ such that:

(i) The set $A_0$ contains $m - 1$ elements, and each set $A_1, \ldots, A_{n-1}$ contains $m$ elements.

(ii) Two critical points of $B$ have the same multiplicity whenever they belong to the same set $A_k$ for some $k = 1, \ldots, n - 1$.

(iii) Let $f_0$ be (one and then any) Blaschke product of degree $m$ with $A_0$ as set of critical points. Then $f_0$ is constant on each $A_k$ for $k = 1, \ldots, n - 1$.

If these conditions are satisfied then $B$ can be decomposed as $B = g_0 \circ f_0$, and the general form of such decompositions is

$$B = (g_0 \circ h^{-1}) \circ (h \circ f_0)$$

with a conformal disk automorphism $h$. 
Checking the Conditions in the DGSSV-Theorem

(i) The partitioning of critical points can be read off from the color and the shape of the exceptional tiles.
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(ii) All critical points have multiplicity 1.

(iii) How do we see that $f_0$ is constant on each set $A_k$?
Condition (iii) is equivalent to the fact that $f_0$ maps all exceptional tiles associated with the same set $A_k$ onto one and the same tile. This is a matter of symmetry.
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Knowing (or guessing) which exceptional tiles contain the critical points of \( f_0 \), this can be checked by constructing symmetric paths that connect the tiles in the corresponding set (as shown for the yellow tiles).
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All this holds up to some error depending on the resolution of the tiling.
A Theorem of Ritt

There is another, more abstract, criterion for decomposability of Blaschke products (originally stated for polynomials).

**Theorem (Ritt, 1922)**

A (normalized) Blaschke product is decomposable if and only if its monodromy group acts *imprimitively* on the sheets of its Riemann surface.

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A group $G$ operating on a set $S$ acts imprimitively, if there is a non-trivial partition of $S$ into (disjoint) subsets $P_1, \ldots, P_m$ which is respected by $G$, i.e., if $s_1, s_2 \in P_k$ and $g \in G$, then $g(s_1), g(s_2) \in P_j$ for some $j$. 
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Can this partition be seen in the phase plot of a decomposable Blaschke product?
This is a phase plot of $B = g \circ f$ with $m = \deg f = 3$ and $n = \deg g = 3$. Its monodromy group $M_B$ acts on the sheets of the Riemann surface $S_B$. 
Visualizing Ritt’s Theorem

This is a phase plot of $B = g \circ f$ with $m = \deg f = 3$ and $n = \deg g = 3$. Its monodromy group $M_B$ acts on the sheets of the Riemann surface $S_B$. The sheets are associated with the (basins of) the zeros.
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Here are the generators of $M_B$. These three cells together represent one generator, these represent another one, this single cell is the third one, and this is the last one.
Visualizing Ritt’s Theorem

This is a phase plot of $B = g \circ f$ with $m = \deg f = 3$ and $n = \deg g = 3$. Its monodromy group $M_B$ acts on the sheets of the Riemann surface $S_B$. The last two are associated with critical points of $f$. 
This is a phase plot of \( B = g \circ f \) with \( m = \deg f = 3 \) and \( n = \deg g = 3 \). Its monodromy group \( M_B \) acts on the sheets of the Riemann surface \( S_B \).

The last two are associated with critical points of \( f \), these act on \( m = 3 \) zeros (sheets) of \( B \).
Visualizing Ritt’s Theorem

This is a phase plot of \( B = g \circ f \) with \( m = \deg f = 3 \) and \( n = \deg g = 3 \). Its **monodromy group** \( M_B \) acts on the **sheets** of the Riemann surface \( S_B \).

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The last two are associated with critical points of $f$, these act on $m = 3$ zeros (sheets) of $B$ (this is a non-trivial fact which follows from the transitivity of the monodromy group of $f$.) These zeros form the first group of the partition.
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This is a phase plot of $B = g \circ f$ with $m = \text{deg } f = 3$ and $n = \text{deg } g = 3$. Its monodromy group $M_B$ acts on the sheets of the Riemann surface $S_B$. The last two are associated with critical points of $f$, these act on $m = 3$ zeros (sheets) of $B$ (this is a non-trivial fact which follows from the transitivity of the monodromy group of $f$.) These zeros form the first group of the partition. Applying the remaining $n - 1$ generators successively to the $m$ zeros in the first group, yields the members of the other groups.
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Applying the remaining $n - 1$ generators successively to the $m$ zeros in the first group, yields the members of the other groups.
Visualizing Ritt’s Theorem

This is a phase plot of $B = g \circ f$ with $m = \deg f = 3$ and $n = \deg g = 3$. Its monodromy group $M_B$ acts on the sheets of the Riemann surface $S_B$. The generators of $M_B$ associated with critical points of $f$ respect the partition, since they act only inside the first (yellow) group.
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This is a phase plot of $B = g \circ f$ with $m = \deg f = 3$ and $n = \deg g = 3$. Its monodromy group $M_B$ acts on the sheets of the Riemann surface $S_B$. There is somewhat more to discover.
Visualizing Ritt’s Theorem

This is a phase plot of $B = g \circ f$ with $m = \deg f = 3$ and $n = \deg g = 3$. Its monodromy group $M_B$ acts on the sheets of the Riemann surface $S_B$.

There is somewhat more to discover.

Each of the highlighted superbasins is mapped by $f$ onto a copy of the unit disk. The generators of associated with $f$ permute these.
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The monodromy group of a composition

\[ f \circ g \]
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\[ f \circ g \circ f \]
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\[ g \circ f \]
The monodromy group of a composition

\[ f \circ g \]
The monodromy group of a composition \( f \circ g \) is the direct product of the monodromy groups of its factors.
A Picture Book of Functions
A picture book of complex functions

The exponential function

\[ e^z = 1 + \frac{z}{1!} + \frac{z^2}{2!} + \ldots + \frac{z^k}{k!} + \ldots \]
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The sine function is a sum of two exponentials,

\[ \sin z = \frac{1}{2i} \left( e^{iz} - e^{-iz} \right) \]
A picture book of complex functions

The exponential function

\[ e^z = 1 + \frac{z}{1!} + \frac{z^2}{2!} + \ldots + \frac{z^k}{k!} + \ldots \]

A linear combination of three exponential functions,

\[ f(z) = \sum c_k e^{a_k z} \]
A finite Blaschke product

\[ f(z) := \prod_{k=1}^{50} \frac{z - z_k}{1 - \overline{z_k} z}. \]

Wilhelm Blaschke (1885-1962)
A finite Blaschke product on the Riemann sphere

\[ f(z) := \prod_{k=1}^{50} \frac{z - z_k}{1 - \overline{z_k}z}. \]

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An infinite Blaschke product on the Riemann sphere

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A singular inner function generated from an atomic measure at the fifth roots of unity,

\[ f(z) = \prod_{k=1}^{5} \exp \frac{z + z_k}{z - z_k}, \]

where \( z_k = \omega^k \) with \( \omega = e^{2\pi i/5} \).
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\[ f(z) = \prod_{k=1}^{5} \exp \frac{z + z_k}{z - z_k}, \]

where \( z_k = \omega^k \) with \( \omega = e^{2\pi i/5} \).

This function has no zeros in the unit disk and constant modulus 1 almost everywhere on the unit circle.
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\[
\frac{z}{1-z} + \frac{z^2}{1-z^2} + \ldots + \frac{z^n}{1-z^n} + \ldots
\]

Johann Heinrich Lambert (1728-1777)

The Lambert function is the generating function of the divisor function \( \sigma_0 \), its \( n \)th Taylor coefficient coincides with the number of divisors of \( n \).
A picture book of complex functions

A Jacobi Theta function

\[ f(z) := \sum_{k=-\infty}^{\infty} q^{\frac{k^2}{2}} e^{2\pi i k z} \]

Carl Gustav Jacobi
(1804-1851)
A Weierstrass’ \( \wp \)-Function

\[
f(z) = \frac{1}{z^2} + \sum_{p \in P, p \neq 0} \left[ \frac{1}{(z - p)^2} - \frac{1}{p^2} \right]
\]

Karl Weierstraß (1815-1897)
A picture book of complex functions

A Weierstrass’ \( \wp \)–Function

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\]

and its derivative.

Karl Weierstraß (1815-1897)
The Eisenstein series $G_4$

Eisenstein series

$$G_k(z) = \sum_{(c,d) \in \mathbb{Z} \times \mathbb{Z}, (c,d) \neq (0,0)} \frac{1}{(cz + d)^k}$$

Ferdinand Eisenstein (1823 – 1852)
The Eisenstein series $G_6$

Eisenstein series

$$G_k(z) = \sum_{(c,d) \in \mathbb{Z} \times \mathbb{Z}} \frac{1}{(cz + d)^k}$$

Ferdinand Eisenstein (1823 – 1852)
Klein’s automorphic $j$-function

$$j(z) = 12^3 \frac{20 G_4^3}{20 G_4^3 - 49 G_6^2}$$

Felix Klein (1849-1929)
A picture book of complex functions

Ramanujan's continued fraction, convergent 200.

\[ 1 + \frac{z}{1 + \frac{z^2}{1 + \frac{z^3}{1 + \cdots}}} \]

Srinivasa Ramanujan (1887-1920)
Ramanujan’s continued fraction, convergent 201.

\[ 1 + \frac{z}{1 + \frac{z^2}{1 + \frac{z^3}{1 + \cdots}}} \]

Srinivasa Ramanujan (1887-1920)
A picture book of complex functions

The Riemann Zeta function

\[ f(z) = \frac{1}{1^z} + \frac{1}{2^z} + \frac{1}{3^z} + \frac{1}{4^z} \ldots \]

Bernhard Riemann (1826-1866)
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$$2 \sin(\pi z) \Gamma(z) \zeta(z) = i \oint_C \frac{(-x)^{z-1}}{e^x - 1} \, dx.$$ 

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Moreover, he heuristically estimated the number \( N(T) \) of non-trivial zeros which satisfy \( 0 < \text{Im} z < T \) by

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*Indeed one finds about as many real roots within these bounds, and it is very likely that all roots are real. A strict proof of this fact would be desirable, however, after some unsuccessful attempts, I abandoned searching for one, because it was expendable for the next purpose of my investigations.*
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Though it is known today that more than $10^{14}$ non-trivial zeros indeed lie on the critical line, the problem withstands all attacks and seems far from being solved.
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The Zeta function in the critical strip

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In order to explore this region further we send out scouts.
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Universality of the Zeta function

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The necessity of the condition $\text{chrom } S = 0$ is equivalent to the Riemann Hypothesis.
Software: the complex function explorer

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